RIESZ TRANSFORMS AND MULTIPLIERS FOR THE BESSEL-GRUSHIN OPERATOR

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Abstract. We call the Bessel-Grushin operator of order α to the differential operator

$$G_\alpha = -\partial_x^2 + \frac{\alpha^2 - 1/4}{x^2} - x^2\partial_t^2 \text{ on } (0,\infty)\times\mathbb{R}.$$

We study L^p-boundedness properties of Riesz transforms associated to G_α. Also, we establish that the spectral multiplier $m(G_\alpha)$ is of weak type $(1,1)$ provided that m is in a suitable local Sobolev space. In order to do this we prove weighted Plancherel estimates involving Bessel-Grushin operators.

1. Introduction

The Grushin operator is defined by

$$G = -\Delta - |x|^2\partial_t^2, \quad \text{on } \mathbb{R}^n \times \mathbb{R}.$$

In this paper we consider the operators we call Bessel-Grushin operators which appear when the Laplacian operator in G is replaced by the Bessel operator

$$B_\alpha = \frac{d^2}{dx^2} - \frac{\alpha^2 - 1/4}{x^2}, \quad \alpha > -1/2.$$

We define the Bessel-Grushin operator G_α by

$$G_\alpha = -\partial_x^2 + \frac{\alpha^2 - 1/4}{x^2} - x^2\partial_t^2, \text{ on } (0,\infty)\times\mathbb{R}, \text{ for every } \alpha > -1/2.$$

Our objective in this paper is to study L^p-boundedness properties of Riesz transforms and spectral multipliers associated with G_α. We are motivated by the recent papers of Chen and Sikora [5], Jotsaroop, Sanjay and Thangavelu [15], Martini and Muller [18] and Martini and Sikora [19], about Grushin operators.

We consider the Laguerre operator

$$L_\alpha = -\frac{d^2}{dx^2} + \frac{\alpha^2 - 1/4}{x^2} + x^2, \text{ on } (0,\infty).$$

We have that, for every $k \in \mathbb{N}$,

$$L_\alpha \varphi_k^\alpha = 2(2k + \alpha + 1)\varphi_k^\alpha,$$

where

$$\varphi_k^\alpha(x) = \left(\frac{2\Gamma(k+1)}{\Gamma(k+\alpha+1)}\right)^{1/2} e^{-x^2/2} x^{\alpha+1/2} l_k^\alpha(x^2), \quad x \in (0,\infty),$$

and l_k^α represents the k-th Laguerre polynomial of order α ([28, p. 100-102]). The family $\{\varphi_k^\alpha\}_{k\in\mathbb{N}}$ is an orthonormal basis in $L^2(0,\infty)$.

We denote by $\mathcal{F}_2(f)$ the Fourier transform of $f \in L^2((0,\infty)\times\mathbb{R})$ with respect to the second variable.

The Bessel-Grushin operator $\widetilde{G}_\alpha$ is defined by

$$\widetilde{G}_\alpha f = \mathcal{F}_2^{-1}\left[\sum_{k=0}^\infty 2(2k+\alpha+1)|u| c_k^\alpha(\mathcal{F}_2(f); u)\varphi_k^\alpha(x;u)\right], \quad f \in D(\widetilde{G}_\alpha),$$

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where

$$D(\widetilde{G}_\alpha) = \left\{ f \in L^2((0,\infty)\times\mathbb{R}) : \sum_{k=0}^{\infty} 2(2k+\alpha+1)|u| c_k^\alpha(\mathcal{F}_2(f);u)\varphi_k^\alpha(x;u) \in L^2((0,\infty)\times\mathbb{R}) \right\}.$$

Here, for every $k \in \mathbb{N}$ and $g \in L^2((0,\infty)\times\mathbb{R})$,

$$c_k^\alpha(g;u) = \int_0^\infty \varphi_k^\alpha(x;u) g(x,u)dx, \quad u \in \mathbb{R},$$

where

$$\varphi_k^\alpha(x;u) = \sqrt[4]{|u|}\varphi_k^\alpha(\sqrt{|u|}x), \;\; x \in (0,\infty) \text{ and } u \in \mathbb{R}.$$

Note that, for every $u \in \mathbb{R}\setminus\{0\}$, $\{\varphi_k^\alpha(\cdot,u)\}_{k\in\mathbb{N}}$ is an orthonormal basis in $L^2(0,\infty)$. Moreover, denoting by $L_\alpha(u)$ the operator

$$L_\alpha(u) = -\frac{d^2}{dx^2} + \frac{\alpha^2 - 1/4}{x^2} + u^2x^2, \quad u \in \mathbb{R}\setminus\{0\},$$

we have that

$$L_\alpha(u)\varphi_k^\alpha(\cdot,u) = 2(2k+\alpha+1)|u|\varphi_k^\alpha(\cdot,u), \quad k \in \mathbb{N} \text{ and } u \in \mathbb{R}\setminus\{0\}.$$

If $f \in C_c^\infty((0,\infty)\times\mathbb{R})$, the space of smooth functions with compact support in $(0,\infty)\times\mathbb{R}$, it is clear that $\widetilde{G}_\alpha f = G_\alpha f$. In the sequel we write also G_α to refer us to the operator $\widetilde{G}_\alpha$.

Suppose that m is a bounded Borel function on $(0,\infty)$. We define the spectral multiplier $m(G_\alpha)$ associated with m by

$$m(G_\alpha)f = \mathcal{F}_2^{-1}\left[\sum_{k=0}^\infty m(2(2k+\alpha+1)|u|)c_k^\alpha(\mathcal{F}_2(f);u)\varphi_k^\alpha(x;u)\right], \quad f \in L^2((0,\infty)\times\mathbb{R}).$$

Since m is bounded the operator $m(G_\alpha)$ is bounded from $L^2((0,\infty)\times\mathbb{R})$ into itself. As usual, the question is to give conditions on the function m such that the operator $m(G_\alpha)$ can be extended from $L^2((0,\infty)\times\mathbb{R})\cap L^p((0,\infty)\times\mathbb{R})$ to $L^p((0,\infty)\times\mathbb{R})$ as a bounded operator from $L^p((0,\infty)\times\mathbb{R})$ into itself, when $p \neq 2$.

In the classical Hörmander multiplier, local Sobolev norms are considered to describe smoothness of m in order to get L^p-boundedness of the multiplier operator. These arguments have been used by Christ [6], Duong, Ouhabaz and Sikora [8], Duong, Sikora and Yen [9], Hebisch [14], Hulanicki and Stein [11, cf.], in different settings.

If $1 \le q \le \infty$ and $s > 0$, we denote by $W_q^s(\mathbb{R})$ the L^q-Sobolev space of order s. We choose $\eta \in C_c^\infty(0,\infty)$ not identically zero. The "local" W_q^s norm $\|\cdot\|_{MW_q^s}$ is defined by

$$\|m\|_{MW_q^s} = \sup_{t>0}\|\eta\delta_t m\|_{W_q^s},$$

where $\delta_t m(s) = m(ts),\ t,s \in (0,\infty)$. When we consider different functions η we get equivalent local Sobolev norms.

We now establish our result about spectral multipliers for Bessel-Grushin operators.

Theorem 1.1. *Let $\alpha \ge 1/2$. Suppose that m is a bounded Borel measurable function on $(0,\infty)$ such that $\|m\|_{MW_2^s} < \infty$ for $s > 1$. Then, the spectral multiplier $m(G_\alpha)$ can be extended from $L^1((0,\infty)\times\mathbb{R})\cap L^2((0,\infty)\times\mathbb{R})$ to $L^1((0,\infty)\times\mathbb{R})$ as a bounded operator from $L^1((0,\infty)\times\mathbb{R})$ into $L^{1,\infty}((0,\infty)\times\mathbb{R})$.*

Note that when m is a bounded measurable function on $(0,\infty)$, $m(G_\alpha)$ is bounded from $L^2((0,\infty)\times\mathbb{R})$ into itself, and then classical interpolation theorems and Theorem 1.1 imply that the operator $m(G_\alpha)$ can be extended from $L^p((0,\infty)\times\mathbb{R})\cap L^2((0,\infty)\times\mathbb{R})$ to $L^p((0,\infty)\times\mathbb{R})$ as a bounded operator from $L^p((0,\infty)\times\mathbb{R})$ into it self, for every $1 < p < \infty$, provided that m satisfies the conditions in Theorem 1.1.

The key result in the proof of Theorem 1.1 is a weighted Plancherel type estimate.

Next we introduce Riesz transforms associated with Bessel-Grushin operators.

Let $\alpha > -1/2$ and $u \in \mathbb{R}\setminus\{0\}$. We define

$$A_\alpha(u) = \frac{d}{dx} + |u|x - \frac{\alpha+1/2}{x} \;\text{ and }\; A_\alpha^*(u) = -\frac{d}{dx} + |u|x - \frac{\alpha+1/2}{x}.$$

Note that $A_\alpha^*(u)$ is the "formal" adjoint of $A_\alpha(u)$ in $L^2(0,\infty)$. We have that

$$L_\alpha(u) = A_\alpha^*(u)A_\alpha(u) + 2|u|(\alpha+1).$$

This decomposition suggests to "formally" define the Riesz transforms for the scaled Laguerre operator $L_\alpha(u)$ as follows

$$R_\alpha(u) = A_\alpha(u)L_\alpha^{-1/2}(u) \qquad \text{and} \qquad \widetilde{R}_\alpha(u) = A_\alpha^*(u)L_\alpha^{-1/2}(u).$$

According to some well-known properties of Laguerre functions (see, for instance, [12, (2.17) and (2.18), p. 1004] and [23, p. 406]) we have that

$$A_\alpha(u)\varphi_k^\alpha(x;u) = -2\sqrt{k|u|}\varphi_{k-1}^{\alpha+1}(x;u), \;\; x \in (0,\infty) \text{ and } k \in \mathbb{N},$$

and

$$A_\alpha^*(u)\varphi_k^\alpha(x;u) = -2\sqrt{(k+1)|u|}\varphi_{k+1}^{\alpha-1}(x;u), \;\; x \in (0,\infty) \text{ and } k \in \mathbb{N}.$$

Here, we understand $\varphi_{-1}^\alpha = 0$.

For every $\beta > 0$, the $-\beta$-th power $L_\alpha^{-\beta}(u)$ is defined by

$$L_\alpha^{-\beta}(u)f = \sum_{k\in\mathbb{N}} \frac{c_k^\alpha(u)(f)}{(2(2k+\alpha+1)|u|)^\beta}\varphi_k^\alpha(\cdot,u), \;\; f \in L^2(0,\infty),$$

where, for every $k \in \mathbb{N}$, $c_k^\alpha(u)(f) = \displaystyle\int_0^\infty \varphi_k^\alpha(x;u)f(x)dx$.

We define Riesz transforms on $L^2(0,\infty)$ as follows

$$R_\alpha(u)f = -2\sum_{k=1}^\infty \sqrt{k}\frac{c_k^\alpha(u)(f)}{\sqrt{2(2k+\alpha+1)}}\varphi_{k-1}^{\alpha+1}(\cdot;u), \;\; f \in L^2(0,\infty),$$

and

$$\widetilde{R}_\alpha(u)f = -2\sum_{k=0}^\infty \sqrt{(k+1)}\frac{c_k^\alpha(u)(f)}{\sqrt{2(2k+\alpha+1)}}\varphi_{k+1}^{\alpha-1}(\cdot;u), \;\; f \in L^2(0,\infty).$$

Note that, in virtue of Plancherel equality for Laguerre function spaces, we deduce that $R_\alpha(u)$ and $\widetilde{R}_\alpha(u)$ are bounded operators from $L^2(0,\infty)$ into itself. Moreover, if $f \in span\{\varphi_k^\alpha(\cdot,u)\}_{k\in\mathbb{N}}$, the linear space generated by $\{\varphi_k^\alpha(\cdot,u)\}_{k\in\mathbb{N}}$, then $R_\alpha(u)f = A_\alpha(u)L_\alpha^{-1/2}(u)f$ and $\widetilde{R}_\alpha(u)f = A_\alpha^*(u)L_\alpha^{-1/2}(u)f$. L^p-boundedness properties of Riesz transforms associated with Laguerre function expansions have been established in [12] and [23], among others.

The above comments suggest to define Riesz transforms R_α and $\widetilde{R}_\alpha$ in Bessel-Grushin settings as follows

$$R_\alpha(f)(x,t) = \mathcal{F}_2^{-1}(R_\alpha(u)(\mathcal{F}_2(f))(x,u))(t),$$

and

$$\widetilde{R}_\alpha(f)(x,t) = \mathcal{F}_2^{-1}(\widetilde{R}_\alpha(u)(\mathcal{F}_2(f))(x,u))(t),$$

for every $f \in L^2((0,\infty)\times\mathbb{R})$. Then, Plancherel theorem for Fourier transform implies that R_α and $\widetilde{R}_\alpha$ are bounded operators from $L^2((0,\infty)\times\mathbb{R})$ into itself.

We prove the following result.

Theorem 1.2. *Let $\alpha \geq 1/2$ and $1 < p < \infty$. Then, the Riesz transforms R_α and $\widetilde{R}_\alpha$ are bounded operators from $L^p((0,\infty)\times\mathbb{R})$ into itself.*

In order to prove this theorem we start using the main idea in the proof of [15, Theorem 1.1], namely, we see R_α and $\widetilde{R}_\alpha$ as Banach valued Fourier multipliers and then we use the celebrated Weis' multiplier result [31, Theorem 3.4]. But to show that the R-boundedness properties hold for the family of operators which appear in the Bessel-Grushin context, we can not proceed as in [15, Section 2] because Laguerre functions have not as nice operational properties as Hermite functions. Roughly speaking we take advantage that the operators we need to study are bounded perturbations of those operator handled in [15].

In the following sections we prove Theorems 1.1 and 1.2.

Throughout this paper we always denote by c and C positive constants that can change from one line to the other.

2. Proof of Theorem 1.1

The strategy of the proof of this theorem is the same as in [19] (see also [5], [18] and [26]) and the key result is a weighted Plancherel inequality. Laguerre expansions play an important role and we need to get estimations involving Laguerre functions.

The Bessel-Grushin operator G_α is selfadjoint and positive in $L^2((0,\infty)\times\mathbb{R})$. Then, $-G_\alpha$ generates a semigroup of contractions $\{e^{-tG_\alpha}\}_{t>0}$ in $L^2((0,\infty)\times\mathbb{R})$. Moreover, since $G_\alpha - G = (\alpha^2-1/4)/x^2$, by using the perturbation formula (see [10, Corollary 1.7, p. 161]) we get

$$e^{-tG}f - e^{-tG_\alpha}f = \int_0^t e^{-(t-s)G}\,\frac{\alpha^2-1/4}{x^2}\,e^{-sG_\alpha}f\,ds. \tag{1}$$

Here and in the sequel, we identify each measurable function f on $(0,\infty)\times\mathbb{R}$ with the function f_0 defined by

$$f_0(x,t) = \begin{cases} f(x,t), & x>0,\\ 0, & x\leq 0. \end{cases}, \quad t\in\mathbb{R}.$$

>From (1) we deduce that $e^{-tG_\alpha}f \leq e^{-tG}f$, $0\leq f\in L^2((0,\infty)\times\mathbb{R})$ and $t>0$.

According to [19, Proposition 3] there exists a distance ρ in $\mathbb{R}^2$ such that the triple $(\mathbb{R}^2,\rho,|\cdot|)$, where $|\cdot|$ denotes the Lebesgue measure in $\mathbb{R}^2$, is a homogeneous type space (in the sense of Coifman and Weiss [7]), and that

$$0\leq \mathbb{W}_t((x_1,t_1),(x_2,t_2)) \leq C\frac{e^{-\rho((x_1,t_1),(x_2,t_2))^2/t}}{|B_\rho((x_2,t_2),\sqrt{t})|}, \quad (x_j,t_j)\in\mathbb{R}^2,\ j=1,2, \text{ and } t>0,$$

where, for every $t>0$, $\mathbb{W}_t$ represents the integral kernel of e^{-tG}.

Hence, for every $t>0$, the operator e^{-tG_α} is bounded from $L^1((0,\infty)\times\mathbb{R})$ into $L^q((0,\infty)\times\mathbb{R})$, $1\leq q<\infty$. Then, for every $t>0$,

$$e^{-tG_\alpha}(f)(x_1,t_1) = \int_{(0,\infty)\times\mathbb{R}} \mathbb{W}_t^\alpha((x_1,t_1),(x_2,t_2))f(x_2,t_2)dx_2dt_2, \quad f\in L^2((0,\infty)\times\mathbb{R}), \tag{2}$$

and

$$0\leq \mathbb{W}_t^\alpha((x_1,t_1),(x_2,t_2)) \leq C\frac{e^{-\rho((x_1,t_1),(x_2,t_2))^2/t}}{|B_\rho((x_2,t_2),\sqrt{t})|}, \quad (x_j,t_j)\in(0,\infty)\times\mathbb{R},\ j=1,2, \text{and } t>0.$$

By defining e^{-tG_α}, $t>0$, by (2) on $L^p((0,\infty)\times\mathbb{R})$, $\{e^{-tG_\alpha}\}_{t>0}$ is a bounded semigroup on $L^p((0,\infty)\times\mathbb{R})$, for every $1\leq p<\infty$.

Moreover, by [25, Propisition 1.4] the semigroup $\{e^{-tG_\alpha}\}_{t>0}$ is bounded holomorphic in $L^2((0,\infty)\times\mathbb{R})$ with angle $\pi/2$. According to [25, Theorem 2.4], for every $(x_j,t_j)\in(0,\infty)\times\mathbb{R}$, $j=1,2$, the integral kernel $\mathbb{W}_t^\alpha((x_1,t_1),(x_2,t_2))$, $t>0$, can be extended to an holomorphic function $\mathbb{W}_z^\alpha((x_1,t_1),(x_2,t_2))$, $Re(z)>0$.

By proceeding as in the proof of [8, Lemmas 2.1 and 4.1] and [26, Lemma 3.3], for instance, we can show the following properties of the integral heat kernel $\mathbb{W}_z^\alpha$, $Re(z)>0$.

Lemma 2.1. *Let $\alpha\geq 1/2$. Then,*

(a) *Fort every $(x_1,t_1)\in(0,\infty)\times\mathbb{R}$ and $t,r>0$,*

$$\int_{((0,\infty)\times\mathbb{R})\setminus B_\rho((x_1,t_1),r)} |\mathbb{W}_t^\alpha((x_1,t_1),(x_2,t_2))|^2dx_2dt_2 \leq C\frac{e^{-r^2/t}}{|B_\rho((x_1,t_1),\sqrt{t})|}.$$

and

$$\|\mathbb{W}_t^\alpha((x_1,t_1),\cdot)\|^2_{L^2((0,\infty)\times\mathbb{R})} = \|\mathbb{W}_t^\alpha(\cdot,(x_1,t_1))\|^2_{L^2((0,\infty)\times\mathbb{R})} \leq \frac{C}{|B_\rho((x_1,t_1),\sqrt{t})|}.$$

(b) *For every $s>0$, $\tau\in\mathbb{R}$, $R>0$ and $(x_1,t_1)\in(0,\infty)\times\mathbb{R}$,*

$$\int_{(0,\infty)\times\mathbb{R}} |\mathbb{W}^\alpha_{(1+i\tau)R^{-2}}((x_1,t_1),(x_2,t_2))|^2[\rho((x_1,t_1),(x_2,t_2))]^s dx_2dt_2 \leq C\frac{R^{-s}(1+|\tau|)^s}{|B_\rho((x_1,t_1),1/R)|}.$$

Let $R>0$. As it was commented above the operator $e^{-R^{-2}G_\alpha}$ is bounded from $L^1((0,\infty)\times\mathbb{R})$ into $L^2((0,\infty)\times\mathbb{R})$. Suppose that F is a bounded measurable function on $\mathbb{R}$ such that *supp* $F\subset[0,R^2]$. We define $H_2(\lambda)=e^{-\lambda R^{-2}}$, $\lambda\in\mathbb{R}$, and $H_1=F/H_2$. It is clear that $H_2(G_\alpha)=e^{-R^{-2}G_\alpha}$ and that $H_1(G_\alpha)$ is a bounded operator from $L^2((0,\infty)\times\mathbb{R})$ into itself with

$$\|H_1(G_\alpha)\|_{L^2((0,\infty)\times\mathbb{R})\to L^2((0,\infty)\times\mathbb{R})} \leq \|H_1\|_{L^\infty(0,\infty)} \leq e\|F\|_{L^\infty(0,\infty)}.$$

Hence, the operator $F(G_\alpha)$ is associated to the kernel

$$K_{F(G_\alpha)}((x_1,t_1),(x_2,t_2)) = H_1(G_\alpha)[\mathbb{W}^\alpha_{R^{-2}}(\cdot,(x_2,t_2))](x_1,t_1), \quad (x_j,t_j) \in (0,\infty)\times\mathbb{R}, \ \ j=1,2.$$

Then, Lemma 2.1, (a), leads to

$$\|K_{F(G_\alpha)}((x_1,t_1),\cdot)\|^2_{L^2((0,\infty)\times\mathbb{R})} \leq C\frac{\|F\|_{L^\infty(0,\infty)}}{|B_\rho((x_1,t_1),1/R)|}.$$

The arguments presented in the proof of [26, Lemma 3.5] (see also [8, Lemma 4.3, (a)]) allow us to establish the following result.

Lemma 2.2. *Let $\alpha > -1/2$ and $R, s > 0$. For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$\int_{(0,\infty)\times\mathbb{R}} |K_{F(G_\alpha)}((x_1,t_1),(x_2,t_2))|^2[1+R\rho((x_1,t_1),(x_2,t_2))]^s dx_2dt_2 \leq C_\varepsilon \frac{\|\delta_{R^2}F\|^2_{W^\infty_{s/2+\varepsilon}}}{|B_\rho((x_1,t_1),1/R)|},$$

for every bounded measurable function F such that supp $F \subset [0,R^2]$.

We now define the operators

$$Mf(x,t) = xf(x,t), \quad f \in D(M) = \{g \in L^2((0,\infty)\times\mathbb{R}), \ \ xg \in L^2((0,\infty)\times\mathbb{R})\},$$

and

$$\mathcal{D}f = \mathcal{F}_2^{-1}(-iu\mathcal{F}_2(f)(u)), \quad f \in D(\mathcal{D}) = \{g \in L^2((0,\infty)\times\mathbb{R}), \ \ u\mathcal{F}_2(g) \in L^2((0,\infty)\times\mathbb{R})\}.$$

It is clear that M is a positive and selfadjoint operator. Moreover, we have that

$$|\mathcal{D}|f = \mathcal{F}_2^{-1}(|u|\mathcal{F}_2(f)(u)), \quad f \in D(\mathcal{D}).$$

Then, $|\mathcal{D}|^m f = \mathcal{F}_2^{-1}(|u|^m\mathcal{F}_2(f)(u))$, $f \in D(|\mathcal{D}|^m)$, and $m \in \mathbb{N}$. Also we define the operator $M\mathcal{D}$ by

$$\mathcal{S}f(x,t) = x\mathcal{F}_2^{-1}\Big(-iu\mathcal{F}_2\Big(f(x,\cdot)\Big)(u)\Big)(t),$$

for every $f \in D(\mathcal{S}) = \{f \in L^2((0,\infty)\times\mathbb{R}) : xu\mathcal{F}_2(f(x,\cdot))(u) \in L^2((0,\infty)\times\mathbb{R})\}$. Note that $D(M\mathcal{D}) \subsetneq D(\mathcal{S})$.

Next result is a version of [19, Proposition 4] in our setting.

Lemma 2.3. *Let $\alpha > -1/2$ and $\gamma > 0$. Then,*

$$\|M^\gamma f\|_{L^2((0,\infty)\times\mathbb{R})} \leq C\|G_\alpha^{\gamma/2}|\mathcal{D}|^{-\gamma}f\|_{L^2((0,\infty)\times\mathbb{R})}, \tag{3}$$

for every $f \in Ran(|\mathcal{D}|^\gamma)$, the range of $|\mathcal{D}|^\gamma$, such that $|\mathcal{D}|^{-\gamma}f \in D(G_\alpha^{\gamma/2})$.

Proof. Our first objective is to show that, for every $m \in \mathbb{N}$,

$$\|\mathcal{S}^{2m}f\|_{L^2((0,\infty)\times\mathbb{R})} \leq C\|G_\alpha^m f\|_{L^2((0,\infty)\times\mathbb{R})}, \quad f \in D(G_\alpha^m). \tag{4}$$

We consider the operator

$$\mathcal{A}_\alpha(f)(x,t) = \mathcal{F}_2^{-1}\left[\sum_{k=0}^\infty \frac{c_k^\alpha(\mathcal{F}_2(f);u)}{2(2k+\alpha+1)|u|}\varphi_k^\alpha(x,u)\right](t), \quad f \in D(\mathcal{A}_\alpha),$$

being

$$D(\mathcal{A}_\alpha) = \left\{g \in L^2((0,\infty)\times\mathbb{R}) \ : \ \sum_{k=0}^\infty \frac{c_k^\alpha(\mathcal{F}_2(f);u)}{2(2k+\alpha+1)|u|}\varphi_k^\alpha(x,u) \in L^2((0,\infty)\times\mathbb{R})\right\}.$$

Observe that, for every $f \in D(\mathcal{A}_\alpha)$, $\mathcal{A}_\alpha f \in D(G_\alpha)$ and $G_\alpha\mathcal{A}_\alpha f = f$. Furthermore, if $f \in D(G_\alpha)$, $G_\alpha f \in D(\mathcal{A}_\alpha)$ and $\mathcal{A}_\alpha G_\alpha f = f$.

We treat the case $m = 1$, for which inequality in (4) is equivalent to the following one

$$\|\mathcal{S}^2\mathcal{A}_\alpha f\|_{L^2((0,\infty)\times\mathbb{R})} \leq C\|f\|_{L^2((0,\infty)\times\mathbb{R})}, \quad f \in D(\mathcal{A}_\alpha). \tag{5}$$

According to Plancherel equality for the Fourier transform, (5) holds if, and only if, $D(G_\alpha) \subseteq D(\mathcal{S}^2) = \{f \in L^2((0,\infty)\times\mathbb{R}) : x^2u^2\mathcal{F}_2(f(x,\cdot))(u) \in L^2((0,\infty)\times\mathbb{R})\}$ and the operator T_α defined by

$$T_\alpha(f)(x,u) = \sum_{k=0}^\infty \frac{x^2|u|c_k^\alpha(\mathcal{F}_2(f);u)}{2(2k+\alpha+1)}\varphi_k^\alpha(x,u), \quad f \in L^2((0,\infty)\times\mathbb{R})$$

is bounded from $L^2((0,\infty)\times\mathbb{R})$ into itself.

In order to show the L^2-boundedness property for T_α we consider the operator L_α^{-1} defined by

$$L_\alpha^{-1} g = \sum_{k=0}^{\infty} \frac{c_k^\alpha(g)}{2(2k+\alpha+1)} \varphi_k^\alpha, \quad g \in L^2(0,\infty),$$

where

$$c_k^\alpha(g) = \int_0^\infty \varphi_k^\alpha(x) g(x) dx, \quad k \in \mathbb{N}. \tag{6}$$

The operator $x^2 L_\alpha^{-1}$ is bounded from $L^2(0,\infty)$ into itself. Indeed, let $g \in L^2(0,\infty)$. We define the function g_0 by

$$g_0(x) = g(x),\ x \in (0,\infty), \quad \text{and} \quad g_0(x) = 0,\ x \in (-\infty, 0].$$

We have that $L_\alpha^{-1}(|g|)(x) \leq C H^{-1}(|g_0|)$, $x \in (0,\infty)$, where H represents the Hermite operator. Then, the L^2-boundedness of the operator $x^2 L_\alpha^{-1}$ follows from [3, Lemma 3] (see also [4]). We get

$$\begin{aligned}
\|T_\alpha(f)\|^2_{L^2((0,\infty)\times\mathbb{R})} &= \int_{\mathbb{R}} \int_0^\infty \left| \sum_{k=0}^{\infty} \frac{x^2 |u| c_k^\alpha(\mathcal{F}_2(f); u)}{2(2k+\alpha+1)} \varphi_k^\alpha(x,u) \right|^2 dx du \\
&= \int_{\mathbb{R}} \int_0^\infty \left| \sum_{k=0}^{\infty} \frac{x^2 c_k^\alpha\Big(\mathcal{F}_2(f)\Big(y/\sqrt{|u|}, u\Big)\Big)}{2(2k+\alpha+1)} \varphi_k^\alpha(x) \right|^2 dx \frac{du}{\sqrt{|u|}} \\
&= \int_{\mathbb{R}} \int_0^\infty \left| x^2 L_\alpha^{-1}\Big(\mathcal{F}_2(f)\Big(\frac{y}{\sqrt{|u|}}, u\Big)\Big)(x) \right|^2 dx \frac{du}{\sqrt{|u|}} \\
&\leq C \int_{\mathbb{R}} \int_0^\infty \left| \mathcal{F}_2(f)\Big(\frac{y}{\sqrt{|u|}}, u\Big) \right|^2 dy \frac{du}{\sqrt{|u|}} \\
&\leq C \|f\|^2_{L^2((0,\infty)\times\mathbb{R})}, \quad f \in L^2((0,\infty)\times\mathbb{R}).
\end{aligned}$$

Note that $\mathcal{F}_2(f)(\cdot/\sqrt{|u|}, u) \in L^2(0,\infty)$, a.e. $u \in \mathbb{R}$, and then the coefficient c_k^α in the second equality above, which is given by (6), is understood as a function of u. On the other hand, the property $D(G_\alpha) \subseteq D(\mathcal{S}^2)$ can be also deduced from the previous argument.

An inductive procedure allows to show that (4) is true for every $m \in \mathbb{N}$. The imaginary powers of the operators $\mathcal{S}$ and G_α are bounded in $L^2((0,\infty)\times\mathbb{R})$ (see [20, p. 640] or [32, Theorem B]). By using [17, Theorem 11.6.1] and [1, Theorem 4.1.2] we get that there exists $C > 0$ such that

$$\|\mathcal{S}^\gamma f\|_{L^2((0,\infty)\times\mathbb{R})} \leq C\Big(\|f\|_{L^2((0,\infty)\times\mathbb{R})} + \|G_\alpha^{\gamma/2} f\|_{L^2((0,\infty)\times\mathbb{R})}\Big), \quad f \in D(G_\alpha^{\gamma/2}).$$

In the usual way, the homogeneity allows us to obtain that

$$\|\mathcal{S}^\gamma f\|_{L^2((0,\infty)\times\mathbb{R})} \leq C \|G_\alpha^{\gamma/2} f\|_{L^2((0,\infty)\times\mathbb{R})}, \quad f \in D(G_\alpha^{\gamma/2}).$$

Since $|\mathcal{D}|^\gamma$ is an one to one operator, we deduce that (3) holds. □

Lemma 2.4. *Suppose that H is a compactly supported Borel measurable complex function defined on $\mathbb{R}$. For every $f \in C_c^\infty(0,\infty) \otimes C_c^\infty(\mathbb{R})$, we have that*

$$H(G_\alpha) f(x,t) = \int_{\mathbb{R}} \int_0^\infty K_H^\alpha(y,z;x,t) f(y,z) dy dz, \quad x \in (0,\infty) \text{ and } t \in \mathbb{R},$$

being

$$K_H^\alpha(y,z;x,t) = \frac{1}{2\pi} \int_{\mathbb{R}} \sum_{k=0}^{\infty} H\Big(2(2k+\alpha+1)|u|\Big) \varphi_k^\alpha(x;u) \varphi_k^\alpha(y;u) e^{-iu(z-t)} du, \quad x, y \in (0,\infty) \text{ and } z, t \in \mathbb{R}.$$

Moreover,

$$\|K_H^\alpha(y,z;\cdot,\cdot)\|^2_{L^2((0,\infty)\times\mathbb{R})} = \int_{\mathbb{R}} \sum_{k=0}^{\infty} \left| H\Big(2(2k+\alpha+1)|u|\Big) \varphi_k^\alpha(y;u) \right|^2 du, \quad y \in (0,\infty) \text{ and } z \in \mathbb{R}. \tag{7}$$

Proof. We consider $f(x,t)=h(x)g(t)$, where $h\in C_c^\infty(0,\infty)$ and $g\in C_c^\infty(\mathbb{R})$. We can write

$$\begin{aligned}H(G_\alpha)f(x,t)&=\mathcal{F}_2^{-1}\Big[\sum_{k=0}^\infty H\Big(2(2k+\alpha+1)|u|\Big)\int_0^\infty\varphi_k^\alpha(y;u)h(y)dy\mathcal{F}_2(g)(u)\varphi_k^\alpha(x;u)\Big](t)\\&=\frac{1}{2\pi}\int_{\mathbb{R}}e^{iut}\int_0^\infty\sum_{k=0}^\infty H\Big(2(2k+\alpha+1)|u|\Big)\varphi_k^\alpha(x;u)\varphi_k^\alpha(y;u)h(y)dy\int_{\mathbb{R}}e^{-iuz}g(z)dzdu\\&=\int_{\mathbb{R}}\int_0^\infty K_H^\alpha(y,z;x,t)h(y)g(z)dydz,\quad x\in(0,\infty)\text{ and }t\in\mathbb{R}.\end{aligned}$$

Indeed, the interchange of the order of integration can be justified as follows. Since H has bounded support there exists $b>0$ such that $H(2(2k+\alpha+1)|u|)=0$, provided that $2(2k+\alpha+1)|u|>b$, $u\in\mathbb{R}$ and $k\in\mathbb{N}$. Then, $H(2(2k+\alpha+1)|u|)=0$, when $2(\alpha+1)|u|>b$, $u\in\mathbb{R}$ and $k\in\mathbb{N}$. Hence, we have that

$$\begin{aligned}&\int_{\mathbb{R}}\int_0^\infty\int_{\mathbb{R}}|g(z)||h(y)|\sum_{k=0}^\infty|H\Big(2(2k+\alpha+1\Big)|u|)||\varphi_k^\alpha(x;u)||\varphi_k^\alpha(y;u)|dzdydu\\&\quad\le C\int_{-b/(2(\alpha+1))}^{b/(2(\alpha+1))}\int_0^\infty\int_{\mathbb{R}}|g(z)||h(y)|\sum_{k\in\mathbb{N},\,2k+\alpha+1<b/(2|u|)}|\varphi_k^\alpha(x;u)||\varphi_k^\alpha(y;u)|dzdydu\end{aligned}$$

Furthermore, since $|\varphi_k^\alpha(z)|\le C$, $k\in\mathbb{N}$ and $z\in(0,\infty)$, for a certain $C>0$ (see [22, (27)], we get

$$\begin{aligned}&\int_{-b/(2(\alpha+1))}^{b/(2(\alpha+1))}\int_0^\infty\int_{\mathbb{R}}|g(z)||h(y)|\sum_{k\in\mathbb{N},\,2k+\alpha+1<b/(2|u|)}|\varphi_k^\alpha(x;u)||\varphi_k^\alpha(y;u)|dzdydu\\&\quad\le C\|g\|_{L^1(\mathbb{R})}\|h\|_{L^1(0,\infty)}\int_{-b/(2(\alpha+1))}^{b/(2(\alpha+1))}\frac{1}{|u|^{3/4}}\sum_{k\in\mathbb{N},\,2k+\alpha+1<b/(2|u|)}|u|^{5/4}du\\&\quad\le C\|g\|_{L^1(\mathbb{R})}\|h\|_{L^1(0,\infty)}\int_{-b/(2(\alpha+1))}^{b/(2(\alpha+1))}\frac{du}{|u|^{3/4}}\sum_{k\in\mathbb{N}}\frac{1}{k^{5/4}}<\infty.\end{aligned}$$

On the other hand, by using Plancherel equality for Fourier transforms and Laguerre expansions we easily obtain (7). □

For every $R>0$, we define the weight function,

$$w_R((x,t),(y,z))=\min\{R,1/y\}x,\quad x,y\in(0,\infty)\text{ and }t,z\in\mathbb{R}.$$

The following is our crucial weighted Plancherel inequality.

Lemma 2.5. *Assume that $\gamma\in[0,1/2)$ and H is a compactly supported Borel measurable complex function defined on $\mathbb{R}$. Then,*

(8)
$$\|M^\gamma K_H^\alpha(y,z;\cdot,\cdot)\|_{L^2((0,\infty)\times\mathbb{R})}\le C\int_0^\infty|H(u)|^2\min\{u^{-\gamma+1/2},y^{2\gamma-1}\}du,\quad y\in(0,\infty)\text{ and }z\in\mathbb{R}.$$

Particulary, when $\operatorname{supp}H\subset[R^2,4R^2]$, *for some $R>0$, we have that*

(9)
$$\sup_{(y,z)\in(0,\infty)\times\mathbb{R}}|B_\rho((y,z),1/R)|^{1/2}\|w_R((\cdot,\cdot),(y,z))^\gamma K_H^\alpha(y,z;\cdot,\cdot)\|_{L^2((0,\infty)\times\mathbb{R})}\le C\|\delta_{R^2}H\|_{L^2((0,\infty)\times\mathbb{R})},$$

being $C>0$ independent on R.

Proof. We define, for every $n\in\mathbb{N}$,

$$H_n(z)=\chi_{(1/n,n)}(z)H(z),\quad z\in\mathbb{R}.$$

By using monotone convergence theorem it follows that

$$\|K_H^\alpha(y,z;\cdot,\cdot)\|_{L^2((0,\infty)\times\mathbb{R})}=\lim_{n\to\infty}\|K_{H_n}^\alpha(y,z;\cdot,\cdot)\|_{L^2((0,\infty)\times\mathbb{R})},\quad y\in(0,\infty)\text{ and }z\in\mathbb{R}.$$

Let $n\in\mathbb{N}$. Note that

$$\int_{\mathbb{R}}\sum_{k=0}^\infty\Big|H_n\Big(2(2k+\alpha+1)|u|\Big)\varphi_k^\alpha(y;u)\Big|^2\frac{\Big(2(2k+\alpha+1)|u|\Big)^\gamma}{|u|^{2\gamma}}du<\infty$$

if, and only if,

$$\int_{\mathbb{R}} \sum_{k=0}^{\infty} \Big| H_n\Big(2(2k+\alpha+1)|u|\Big) \varphi_k^\alpha(y;u)\Big|^2 \frac{du}{|u|^{2\gamma}} < \infty.$$

Our next objective is to estimate the following function

$$\Lambda_n(y) = \int_{\mathbb{R}} \sum_{k=0}^{\infty} \Big| H_n\Big(2(2k+\alpha+1)|u|\Big) \varphi_k^\alpha(y;u)\Big|^2 \frac{\Big(2(2k+\alpha+1)|u|\Big)^\gamma}{|u|^{2\gamma}} du, \quad y \in (0,\infty).$$

By making straightforward manipulations we get

$$\Lambda_n(y) \leq C \int_0^\infty |H_n(u)|^2 \sum_{k=0}^{\infty} \Big| \varphi_k^\alpha\Big(\frac{\sqrt{u}y}{\sqrt{2(2k+\alpha+1)}}\Big)\Big|^2 \frac{u^{1/2-\gamma}}{(2k+\alpha+1)^{3/2-2\gamma}} du, \quad y \in (0,\infty).$$

According to [21, p. 1124], there exist C, η, λ and $\xi \in (0,\infty)$ such that

$$|\varphi_k^\alpha(x)| \leq C\mathcal{M}_k^\alpha(x), \quad x \in (0,\infty) \text{ and } k \in \mathbb{N},$$

where

$$\mathcal{M}_k^\alpha(x) = x^{\alpha+1/2}\Big(\frac{1}{\nu_k} + x^2\Big)^{-1/4-\alpha/2} (\nu_k^{1/3} + |x^2 - \nu_k|)^{-1/4} \Phi_k^\alpha(x), \quad x \in (0,\infty),$$

and

$$\Phi_k^\alpha(x) = \begin{cases} 1, & 0 \leq x^2 \leq \nu_k, \\ \exp\Big(-\eta|\nu_k - x^2|^{3/2}/\nu_k^{1/2}\Big), & \nu_k \leq x^2 \leq (1+\lambda)\nu_k, \\ e^{-\xi x^2}, & (1+\lambda)\nu_k \leq x^2, \end{cases}$$

being $\nu_k = 4k + 2\alpha + 2$, $k \in \mathbb{N}$.

Then, we deduce that

$$|\varphi_k^\alpha(x)| \leq C \begin{cases} (\nu_k^{1/3} + |x^2 - \nu_k|)^{-1/4}, & x \in (0,\infty), \\ e^{-\xi x^2}, & x^2 \geq (1+\lambda)\nu_k. \end{cases} \tag{10}$$

By proceeding as in the proof of [19, Lemma 9], (10) allows us to obtain that, for every $\varepsilon > 0$,

$$\sup_{x \in (0,\infty)} \sum_{k=0}^{\infty} \frac{\max\{1,x\}^\varepsilon}{(2k+\alpha+1)^{\varepsilon+1/2}} \Big| \varphi_k^\alpha\Big(\frac{x}{\sqrt{2(2k+\alpha+1)}}\Big)\Big|^2 < \infty. \tag{11}$$

From (11) we deduce that

$$\Lambda_n(y) \leq C \int_0^\infty |H_n(u)|^2 \min\{u^{-\gamma+1/2}, y^{2\gamma-1}\} du, \quad y \in (0,\infty),$$

provided that $\gamma \in [0, 1/2)$.

By Lemma 2.3, we get

$$\|M^\gamma K_{H_n}^\alpha(y,z;\cdot,\cdot)\|_{L^2((0,\infty)\times\mathbb{R})} \leq C \int_0^\infty |H_n(u)|^2 \min\{u^{-\gamma+1/2}, y^{2\gamma-1}\} du, \quad y \in (0,\infty) \text{ and } z \in \mathbb{R}.$$

where C does not depend on n. By taking limits as $n \to \infty$ we obtain (8).

Suppose now that $\operatorname{supp} H \subset [R^2, 4R^2]$, where $R > 0$. It is clear that

$$\begin{aligned} \int_0^\infty |H(u)|^2 \min\{u^{-\gamma+1/2}, y^{2\gamma-1}\} du &= \int_1^4 |H(R^2 v)|^2 \min\{R^{-2\gamma+1} v^{-\gamma+1/2}, y^{2\gamma-1}\} R^2 dv \\ &\leq CR^2 \min\{R^{1-2\gamma}, y^{2\gamma-1}\} \int_1^4 |H(R^2 v)|^2 dv, \quad y \in (0,\infty). \end{aligned}$$

Since $|B_\rho((x,t),R)| \sim R^2 \max\{x,R\}$, $x, R \in (0,\infty)$ and $t \in \mathbb{R}$ ([19, Proposition 3, (9)]), we deduce (9). $\square$

The proof of Theorem 1.1 can be finished now by proceeding as in [19, Section 4] (see also [5, Section 4]).

3. Proof of Theorem 1.2

The Laguerre operator $-L_\alpha$ generates the semigroup of contractions $\{W_t^\alpha\}_{t>0}$ in $L^2(0,\infty)$, where, for every $t>0$,

$$W_t^\alpha(g) = \sum_{k=0}^\infty e^{-2t(2k+\alpha+1)} c_k^\alpha(g)\varphi_k^\alpha, \quad g\in L^2(0,\infty),$$

where, for every $k\in\mathbb{N}$,

$$c_k^\alpha(g) = \int_0^\infty \varphi_k^\alpha(y)g(y)dy.$$

According to the Mehler's formula [29, (1.1.47)], for every $t>0$, we can write

$$W_t^\alpha(g)(x) = \int_0^\infty W_t^\alpha(x,y)g(y)dy, \tag{12}$$

for every $g\in L^2(0,\infty)$, where

$$W_t^\alpha(x,y) = \left(\frac{2e^{-2t}}{1-e^{-4t}}\right)^{1/2}\left(\frac{2xye^{-2t}}{1-e^{-4t}}\right)^{1/2} I_\alpha\left(\frac{2xye^{-2t}}{1-e^{-4t}}\right)\exp\left(-\frac{1}{2}(x^2+y^2)\frac{1+e^{-4t}}{1-e^{-4t}}\right),\ t,x,y\in(0,\infty).$$

Here I_α denote the modified Bessel function of the first kind and order α. By defining W_t^α, $t>0$, in $L^p(0,\infty)$, $1\le p\le\infty$, by the integral in (12), $\{W_t^\alpha\}_{t>0}$ is a semigroup of contractions in $L^p(0,\infty)$, $1\le p\le\infty$ (see [24, Theorem 4.1]).

Let $u\in\mathbb{R}\setminus\{0\}$. Straightforward manipulations allow us to show that the operator $-L_\alpha(u)$ generates on $L^p(0,\infty)$, $1\le p<\infty$, the semigroup of operators $\{W_{t,u}^\alpha\}_{t>0}$ where, for every $t>0$,

$$W_{t,u}^\alpha(g)(x) = \int_0^\infty W_t^\alpha(x,y;u)g(y)dy, \quad g\in L^p(0,\infty),\ 1\le p<\infty,$$

being

$$W_t^\alpha(x,y;u) = \sqrt{|u|}W_{t|u|}^\alpha(\sqrt{|u|}x,\sqrt{|u|}y), \quad t,x,y\in(0,\infty).$$

We can write, for every $g\in L^p(0,\infty)$, $1<p<\infty$,

$$L_\alpha(u)^{-1/2}g(x) = \frac{1}{\sqrt{\pi}}\int_0^\infty W_{t,u}^\alpha(g)(x)\frac{dt}{\sqrt{t}}, \quad x\in(0,\infty).$$

By using the arguments given in [2] we can see that, for every $g\in L^2(0,\infty)$,

$$R_\alpha(u)(g)(x) = \lim_{\varepsilon\to0^+}\int_{0,|x-y|>\varepsilon}^\infty R_\alpha(x,y;u)g(y)dy, \quad \text{a.e. } x\in(0,\infty), \tag{13}$$

where

$$R_\alpha(x,y;u) = \frac{1}{\sqrt{\pi}}\int_0^\infty A_\alpha(u)W_t^\alpha(x,y;u)\frac{dt}{\sqrt{t}}, \quad x,y\in(0,\infty),\ x\neq y.$$

Moreover, the operator $R_\alpha(u)$ defined by (13) can be extended from $L^2(0,\infty)\cap L^p(0,\infty)$ to $L^p(0,\infty)$ as a bounded operator from $L^p(0,\infty)$ into itself, for every $1<p<\infty$.

Also, for every $g\in L^2(0,\infty)$, we have that

$$\widetilde{R}_\alpha(u)(g)(x) = \lim_{\varepsilon\to0^+}\int_{0,|x-y|>\varepsilon}^\infty \widetilde{R}_\alpha(x,y;u)g(y)dy, \quad \text{a.e. } x\in(0,\infty), \tag{14}$$

where

$$\widetilde{R}_\alpha(x,y;u) = \frac{1}{\sqrt{\pi}}\int_0^\infty A_\alpha^*(u)W_t^\alpha(x,y;u)\frac{dt}{\sqrt{t}}, \quad x,y\in(0,\infty),\ x\neq y,$$

and the operator $\widetilde{R}_\alpha(u)$ defined by (14) can be extended from $L^2(0,\infty)\cap L^p(0,\infty)$ to $L^p(0,\infty)$ as a bounded operator from $L^p(0,\infty)$ into itself.

In this section we prove that the Riesz transform R_α is bounded from $L^p((0,\infty)\times\mathbb{R})$ into itself, for every $1<p<\infty$. To see this property for $\widetilde{R}_\alpha$ we can proceed in a similar way.

The operator R_α is defined by

$$R_\alpha(f)(x,t) = \mathcal{F}_2^{-1}(R_\alpha(u)(\mathcal{F}_2(f))(x,u))(t), \quad f\in L^2((0,\infty)\times\mathbb{R}).$$

Let $1<p<\infty$. We identify $L^p((0,\infty)\times\mathbb{R})$ with $L^p(\mathbb{R},L^p(0,\infty))$. The Riesz transform can be understood as a Banach valued Fourier multiplier. Indeed, let $f\in C_c^\infty((0,\infty)\times\mathbb{R})\subseteq S(\mathbb{R},L^p(0,\infty))$, where $S(\mathbb{R},L^p(0,\infty))$ denotes the $L^p(0,\infty)$-valued Schwartz functions space. Then, $\mathcal{F}(f)\in S(\mathbb{R},L^p(0,\infty))$;

being $\mathcal{F}$ the ($L^p(0,\infty)$-valued) Fourier transform. By using Plancherel equality for Laguerre functions expansion we obtain

$$\|R_\alpha(u)\|_{L^2(0,\infty)\to L^2(0,\infty)} \le 1, \quad u\in\mathbb{R}\setminus\{0\}. \tag{15}$$

Let $u\in\mathbb{R}\setminus\{0\}$. From (13) we get, for every $g\in C_c^\infty(0,\infty)$,

$$R_\alpha(u)(g)(x)=\int_0^\infty R_\alpha(x,y;u)g(y)dy, \quad \text{a.e. } x\notin supp\ g.$$

Moreover, we have that

$$\begin{aligned} A_\alpha(u)W_t^\alpha(x,y;u) &= |u|\left(\frac{d}{d(\sqrt{|u|}x)}+\sqrt{|u|}x-\frac{\alpha+1/2}{\sqrt{|u|}x}\right)\left[W_{t|u|}^\alpha(\sqrt{|u|}x,\sqrt{|u|}y)\right]\\ &= |u|\left[A_\alpha(1)W_{t|u|}^\alpha(x_1,y_1)\right]_{|x_1=\sqrt{|u|}x,\ y_1=\sqrt{|u|}y}, \quad x,y\in(0,\infty),\ x\neq y. \end{aligned}$$

Then,

$$\begin{aligned} R_\alpha(x,y;u) &= \frac{|u|}{\sqrt{\pi}}\int_0^\infty \left[A_\alpha(1)W_{t|u|}^\alpha(x_1,y_1)\right]_{|x_1=\sqrt{|u|}x,\ y_1=\sqrt{|u|}y}\frac{dt}{\sqrt{t}}\\ &= \frac{\sqrt{|u|}}{\sqrt{\pi}}\int_0^\infty \left[A_\alpha(1)W_s^\alpha(x_1,y_1)\right]_{|x_1=\sqrt{|u|}x,\ y_1=\sqrt{|u|}y}\frac{ds}{\sqrt{s}}\\ &= \sqrt{|u|}R_\alpha(\sqrt{|u|}x,\sqrt{|u|}y;1), \quad x,y\in(0,\infty),\ x\neq y. \end{aligned}$$

Since $R_\alpha(x,y;1)$ is a standard Calderón-Zygmund kernel [23, Proposition 3.1] we can obtain

$$|R_\alpha(x,y;u)|\le\frac{C}{|x-y|}, \quad x,y\in(0,\infty),\ x\neq y, \tag{16}$$

and

$$|\partial_x R_\alpha(x,y;u)|+|\partial_y R_\alpha(x,y;u)|\le\frac{C}{|x-y|^2}, \quad x,y\in(0,\infty),\ x\neq y, \tag{17}$$

where the constant $C>0$ does not depend on u.

By using Calderón-Zygmund theory we conclude that R_α can be extended from $L^2(0,\infty)\cap L^p(0,\infty)$ to $L^p(0,\infty)$ as a bounded operator from $L^p(0,\infty)$ into itself (as it was said earlier) and

$$\|R_\alpha(u)\|_{L^p(0,\infty)\to L^p(0,\infty)}\le C,$$

where $C>0$ does not depend on u.

We deduce that $R_\alpha(u)(\mathcal{F}(f)(u))\in L^1(\mathbb{R},L^p(0,\infty))$ and then $\mathcal{F}^{-1}(R_\alpha(u)(\mathcal{F}(f)(u)))\in L^\infty(\mathbb{R},L^p(0,\infty))$. Furthermore, we have that

$$R_\alpha(f)=\mathcal{F}^{-1}(R_\alpha(u)(\mathcal{F}(f)(u))). \tag{18}$$

Indeed, suppose that $g\in L^1(\mathbb{R},L^p(0,\infty))$. We understand g as a function defined in $\mathbb{R}\times(0,\infty)$. Let $h\in L^{p'}(0,\infty)$. By using some properties of the Bochner integral and Hölder inequality we get

$$\begin{aligned} \int_0^\infty h(z)\left(\int_{\mathbb{R}} g(y,\cdot)e^{-xyi}dy\right)(z)dz &= \int_{\mathbb{R}}\int_0^\infty h(z)g(y,z)dze^{-xyi}dy\\ &= \int_0^\infty h(z)\int_{\mathbb{R}} g(y,z)e^{-xyi}dydz, \quad x\in\mathbb{R}. \end{aligned}$$

Then, for every $x\in\mathbb{R}$,

$$\left(\int_{\mathbb{R}} g(y,\cdot)e^{-xyi}dy\right)(z)=\int_{\mathbb{R}} g(y,z)e^{-xyi}dy, \quad \text{a.e. } z\in(0,\infty).$$

According to (18), since $L^p(0,\infty)$ is a UMD Banach space, we use [31, Theorem 3.4] to show that R_α defines a bounded operator from $L^p(\mathbb{R},L^p(0,\infty))$ into itself. It is sufficient to see that the families of operators

$$\Big\{R_\alpha(u)\Big\}_{u\in\mathbb{R}\setminus\{0\}} \quad \text{and} \quad \Big\{u\frac{d}{du}R_\alpha(u)\Big\}_{u\in\mathbb{R}\setminus\{0\}}$$

are R-bounded in $L^p(0,\infty)$. It is well-known that if $\{T(u)\}_{u\in\mathbb{R}\setminus\{0\}}$ is a set of bounded operators in $L^p(0,\infty)$, then $\mathcal{T}=\{T(u)\}_{u\in\mathbb{R}\setminus\{0\}}$ is R-bounded in $L^p(0,\infty)$ if, and only if, there exists $C>0$ such that

$$\Big\|\Big(\sum_{j=1}^N |T(u_j)g_j|^2\Big)^{1/2}\Big\|_{L^p(0,\infty)}\le C\Big\|\Big(\sum_{j=1}^N |g_j|^2\Big)^{1/2}\Big\|_{L^p(0,\infty)}, \tag{19}$$

for every sequence $(u_j)_{j=1}^N \subset \mathbb{R}\setminus\{0\}$ and every sequence $(g_j)_{j=1}^N \subset L^p(0,\infty)$ and $N\in\mathbb{N}$.

Suppose that $\{T(u)\}_{u\in\mathbb{R}\setminus\{0\}}$ is a family of bounded operators in $L^2(0,\infty)$ such that

$$\sup_{u\in\mathbb{R}\setminus\{0\}} \|T(u)\|_{\mathcal{L}(L^2(0,\infty))} < \infty,$$

being $\mathcal{L}(L^2(0,\infty))$ the space of bounded linear mappings from $L^2(0,\infty)$ into itself. Moreover, assume that, for every $u\in\mathbb{R}\setminus\{0\}$ and $g\in C_c^\infty(0,\infty)$,

$$T(u)g(x) = \int_0^\infty K_u(x,y)g(y)dy, \quad x\notin \operatorname{supp}(g),$$

where

$$\sup_{u\in\mathbb{R}\setminus\{0\}} |K_u(x,y)| \leq \frac{C}{|x-y|}, \quad x,y\in(0,\infty), x\neq y,$$

and

$$\sup_{u\in\mathbb{R}\setminus\{0\}} \Big(|\partial_x K_u(x,y)| + |\partial_y K_u(x,y)|\Big) \leq \frac{C}{|x-y|^2}, \quad x,y\in(0,\infty), x\neq y.$$

Then, by [30, Theorem 1.3 Chapter XII], (19) holds for every sequences $(u_j)_{j=1}^\infty \subset \mathbb{R}\setminus\{0\}$ and $(g_j)_{j=1}^\infty \subset L^p(0,\infty)$.

Lemma 3.1. *Let $\alpha > -1/2$. The family of operators $\{R_\alpha(u)\}_{u\in\mathbb{R}\setminus\{0\}}$ is R-bounded in $L^p(0,\infty)$.*

Proof. It is enough to take into account the above observation and (13), (15), (16) and (17). $\square$

The R-boundedness of the family $\{u\frac{d}{du}R_\alpha(u)\}_{u\in\mathbb{R}\setminus\{0\}}$ and the differentiability of R_α are more involved. In order to see these last properties we are going to use some results established in [15]. We need to recall some definitions related to the Hermite operator. We denote by H the Hermite operator

$$H = -\frac{d^2}{dx^2} + x^2, \quad \text{on } \mathbb{R}.$$

This operator can be written as follows

$$H = -\frac{1}{2}(AA^* + A^*A),$$

where $A = d/dx + x$ and $A^* = -d/dx + x$. Note that A^* is the "formal" adjoint of A in $L^2(\mathbb{R})$. For every $k\in\mathbb{N}$, the k-th Hermite function h_k is defined by

$$h_k(x) = (\sqrt{\pi}2^k k!)^{-1/2} e^{-x^2/2} H_k(x), \quad x\in\mathbb{R},$$

where H_k denotes the k-th Hermite polynomial [16, (4.9.1) and (4.9.2)]. We have that

$$Hh_k = (2k+1)h_k, \quad k\in\mathbb{N}.$$

The system $\{h_k\}_{k\in\mathbb{N}}$ is an orthonormal basis in $L^2(\mathbb{R})$. The operator $-H$ generates the semigroup of contractions $\{W_t\}_{t>0}$ in $L^2(\mathbb{R})$, being for every $t>0$,

$$W_t(f) = \sum_{k=0}^\infty e^{-t(2k+1)} b_k(f) h_k, \quad f\in L^2(\mathbb{R}),$$

and

$$b_k(f) = \int_{\mathbb{R}} h_k(y) f(y) dy, \quad k\in\mathbb{N}.$$

By using Mehler's formula for Hermite functions [29, (1.1.36)] we obtain

$$W_t(f)(x) = \int_{\mathbb{R}} W_t(x,y) f(y) dy, \quad f\in L^2(\mathbb{R}) \text{ and } t>0, \tag{20}$$

where

$$W_t(x,y) = \frac{1}{\sqrt{\pi}}\left(\frac{e^{-2t}}{1-e^{-4t}}\right)^{1/2} \exp\left(-\frac{1}{4}\left[(x-y)^2\frac{1+e^{-2t}}{1-e^{-2t}} + (x+y)^2\frac{1-e^{-2t}}{1+e^{-2t}}\right]\right), \ x,y\in\mathbb{R} \text{ and } t\in(0,\infty).$$

Moreover, if W_t, $t>0$, is defined by (20), $\{W_t\}_{t>0}$ is a semigroup of contractions in $L^p(\mathbb{R})$, $1\leq p<\infty$.

In order to study Riesz transforms associated with Grushin operator Jotsaroop, Sanjay and Thangavelu [15] considered the scaled Hermite operator $H(u)$ defined by

$$H(u) = -\frac{d^2}{dx^2} + u^2x^2, \quad \text{on } \mathbb{R},$$

for every $u \in \mathbb{R}$. The operator $H(u)$ can be written as

$$H(u) = \frac{1}{2}[A(u)A^*(u) + A^*(u)A(u)],$$

where $A(u) = \frac{d}{dx} + |u|x$ and $A^*(u) = -\frac{d}{dx} + |u|x$, $u \in \mathbb{R}$. Riesz transforms for the operator $H(u)$ were formally defined in [15] by

$$R(u) = A(u)H(u)^{-1/2} \quad \text{and} \quad \widetilde{R}(u) = A^*(u)H(u)^{-1/2}, \quad u \in \mathbb{R} \setminus \{0\}.$$

Here, we only consider the Riesz transform $R(u)$. Let $u \in \mathbb{R} \setminus \{0\}$. By taking in mind [27, (3.1)] the operator $R(u)$ is defined in $L^2(\mathbb{R})$ as follows

$$R(u)(f) = \sum_{k=1}^{\infty} \sqrt{\frac{2k}{2k+1}} b_k(u)(f) h_{k-1}(\cdot; u), \quad f \in L^2(\mathbb{R}),$$

where

$$b_k(u)(f) = \int_{\mathbb{R}} h_k(y; u) f(y) dy, \quad k \in \mathbb{N},$$

and $h_k(x; u) = \sqrt[4]{|u|} h_k(\sqrt{|u|}x)$, $x \in \mathbb{R}$ and $k \in \mathbb{N}$. $R(u)$ is a bounded operator in $L^2(\mathbb{R})$. Moreover, for every $1 < p < \infty$, $R(u)$ can be extended from $L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ as a bounded operator from $L^p(\mathbb{R})$ into itself, and, for every $g \in L^p(\mathbb{R})$,

$$R(u)(g)(x) = \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} R(x, y; u) g(y) dy, \quad \text{a.e. } x \in \mathbb{R},$$

where

$$R(x, y; u) = \frac{1}{\sqrt{\pi}} \int_0^{\infty} A(u) \sqrt{|u|} W_{t|u|}(\sqrt{|u|}x, \sqrt{|u|}y) \frac{dt}{\sqrt{t}}, \quad x, y \in \mathbb{R}.$$

Note that $R(x, y; u) = \sqrt{|u|} R(\sqrt{|u|}x, \sqrt{|u|}y; 1)$, $x, y \in \mathbb{R}$.

Lemma 3.2. *Let $\alpha > 1/2$. The function*

$$\begin{aligned} R_\alpha : \mathbb{R} \setminus \{0\} &\longrightarrow \mathcal{L}(L^p(0,\infty)) \\ u &\longrightarrow R_\alpha(u) \end{aligned}$$

is differentiable.

Proof. In [15] it was proved that the function

$$\begin{aligned} R : \mathbb{R} \setminus \{0\} &\longrightarrow \mathcal{L}(L^p(\mathbb{R})) \\ u &\longrightarrow R(u) \end{aligned}$$

is differentiable. By identifying $g \in L^p(0, \infty)$ with $g_0 \in L^p(\mathbb{R})$ defined by $g_0(x) = g(x)$, $x \in (0, \infty)$, and $g_0(x) = 0$, $x \in (-\infty, 0)$, we have that

$$\begin{aligned} R\ : \mathbb{R} \setminus \{0\} &\longrightarrow \mathcal{L}(L^p(0,\infty)) \\ u &\longrightarrow R(u) \end{aligned}$$

is differentiable. In order to show that R_α is differentiable we will prove that $D_\alpha = R_\alpha - R$ is differentiable.

Henceforth, assume that $u > 0$ and $g \in L^p(0, \infty)$. According to [2, Proposition 3.3] we have that

$$|R_\alpha(x, y; u) - R(x, y; u)| = \sqrt{u} |R_\alpha(\sqrt{u}x, \sqrt{u}y; 1) - R(\sqrt{u}x, \sqrt{u}y; 1)|$$

$$\leq C \begin{cases} \dfrac{1}{x}, & 0 < y \leq \dfrac{x}{2} < \infty, \\[2ex] \dfrac{1}{x}\left(1 + \sqrt{\dfrac{x}{|x-y|}}\right), & 0 < \dfrac{x}{2} < y < 2x < \infty, \\[2ex] \dfrac{1}{y}, & 0 < 2x \leq y < \infty, \end{cases} \tag{21}$$

where $C > 0$ does not depend on u.

Hence,

$$\int_0^{\infty} |R_\alpha(x, y; u) - R(x, y; u)| |g(y)| dy < \infty, \quad x \in (0, \infty),$$

and we can write

$$D_\alpha(u)(g)(x) = \int_0^\infty [R_\alpha(x,y;u) - R(x,y;u)]g(y)dy, \quad x,y \in (0,\infty).$$

We now analyze $\partial_u[R_\alpha(x,y;u) - R(x,y;u)]$, $x,y \in (0,\infty)$. Firstly, we have that

$$\begin{aligned}\partial_u[R_\alpha(x,y;u) - R(x,y;u)] =& \frac{1}{2\sqrt{u}}[R_\alpha(\sqrt{u}x,\sqrt{u}y;1) - R(\sqrt{u}x,\sqrt{u}y;1)] \\ &+ \frac{x}{2}\left[(\partial_x R_\alpha)(\sqrt{u}x,\sqrt{u}y;1) - (\partial_x R)(\sqrt{u}x,\sqrt{u}y;1)\right] \\ &+ \frac{y}{2}\left[(\partial_y R_\alpha)(\sqrt{u}x,\sqrt{u}y;1) - (\partial_y R)(\sqrt{u}x,\sqrt{u}y;1)\right], \quad x,y\in(0,\infty).\end{aligned} \tag{22}$$

In the sequel we will use the following properties of the Bessel function I_α. We have that, for every $z \in (0,\infty)$ [16, (5.7.9) and (5.16.4)]

$$\frac{d}{dz}(z^{-\alpha}I_\alpha(z)) = z^{-\alpha}I_{\alpha+1}(z), \tag{23}$$

$$I_{\alpha+1}(z) = I_{\alpha-1}(z) - \frac{2\alpha}{z}I_\alpha(z), \tag{24}$$

and

$$I_\alpha(z) \sim \frac{z^\alpha}{2^\alpha\Gamma(\alpha+1)}, \quad \text{as } z \to 0^+, \tag{25}$$

Moreover, by [16, (5.11.8)],

$$\sqrt{2\pi z}I_\alpha(z)e^{-z} = \sum_{r=0}^n (-1)^r\frac{[\alpha,r]}{(2z)^r} + \mathcal{O}\Big(\frac{1}{z^{n+1}}\Big), \quad z\in(0,\infty), \tag{26}$$

where $[\alpha,0] = 1$ and

$$[\alpha,r] = \frac{(4\alpha^2-1)(4\alpha^2-3^2)\cdots(4\alpha^2-(2r-1)^2)}{2^{2r}\Gamma(r+1)}, \quad r = 1,2,...$$

To simplify we write

$$a(t) = \frac{2e^{-2t}}{1-e^{-4t}} \quad \text{and} \quad b(t) = \frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}, \quad t>0.$$

By taking into account that

$$A_\alpha(1) = \frac{d}{dx} + x - \frac{\alpha+1/2}{x} = x^{\alpha+1/2}\frac{d}{dx}x^{-\alpha-1/2} + x,$$

and writing $G_t^\alpha(x,y) = A_\alpha(1)W_t^\alpha(x,y)$, $t,x,y\in(0,\infty)$, (23) leads to

$$\begin{aligned}G_t^\alpha(x,y) =& xW_t^\alpha(x,y) + a(t)^{\alpha+1}(xy)^{\alpha+1/2}\partial_x\left[e^{-b(t)(x^2+y^2)}(a(t)xy)^{-\alpha}I_\alpha(a(t)xy)\right] \\ =& a(t)^{\alpha+1}(xy)^{\alpha+1/2}e^{-b(t)(x^2+y^2)} \\ &\times\left[(a(t)xy)^{-\alpha}I_\alpha(a(t)xy)(1-2b(t))x + (a(t)xy)^{-\alpha}I_{\alpha+1}(a(t)xy)ya(t)\right] \\ =& xW_t(x,y)\sqrt{2\pi a(t)xy}I_\alpha(a(t)xy)e^{-a(t)xy} \\ &+ W_t(x,y)\Big[-2b(t)x\sqrt{2\pi a(t)xy}I_\alpha(a(t)xy)e^{-a(t)xy} \\ &+ a(t)y\sqrt{2\pi a(t)xy}I_{\alpha+1}(a(t)xy)e^{-a(t)xy}\Big], \quad t,x,y\in(0,\infty).\end{aligned}$$

Moreover, by defining $G_t(x,y) = A(1)W_t(x,y)$, $t,x,y\in(0,\infty)$, we have that

$$G_t(x,y) = xW_t(x,y) + \Big(a(t)y - 2b(t)x\Big)W_t(x,y), \quad t,x,y\in(0,\infty).$$

Hence,

$$\begin{aligned}G_t(x,y) - G_t^\alpha(x,y) =& xW_t(x,y)\Big(1-\sqrt{2\pi a(t)xy}I_\alpha(a(t)xy)e^{-a(t)xy}\Big) \\ &+ W_t(x,y)\Big[-2b(t)x\Big(1-\sqrt{2\pi a(t)xy}I_\alpha(a(t)xy)e^{-a(t)xy}\Big) \\ &+ a(t)y\Big(1-\sqrt{2\pi a(t)xy}I_{\alpha+1}(a(t)xy)e^{-a(t)xy}\Big)\Big], \quad t,x,y\in(0,\infty).\end{aligned}$$

By using again (23) we get

$$\begin{aligned}
&x\partial_x\Big(G_t(x,y)-G_t^\alpha(x,y)\Big)+y\partial_y\Big(G_t(x,y)-G_t^\alpha(x,y)\Big)\\
&=(1-2b(t))\Big[xW_t(x,y)\Big(1-\sqrt{2\pi a(t)xy}I_\alpha(a(t)xy)e^{-a(t)xy}\Big)\\
&\quad+x^2\partial_xW_t(x,y)\Big(1-\sqrt{2\pi a(t)xy}I_\alpha(a(t)xy)e^{-a(t)xy}\Big)\\
&\quad-x^2W_t(x,y)\partial_x\Big(\sqrt{2\pi a(t)xy}I_\alpha(a(t)xy)e^{-a(t)xy}\Big)\\
&\quad+xy\partial_yW_t(x,y)\Big(1-\sqrt{2\pi a(t)xy}I_\alpha(a(t)xy)e^{-a(t)xy}\Big)\\
&\quad-xyW_t(x,y)\partial_y\Big(\sqrt{2\pi a(t)xy}I_\alpha(a(t)xy)e^{-a(t)xy}\Big)\Big]\\
&\quad+a(t)\Big[xy\partial_xW_t(x,y)\Big(1-\sqrt{2\pi a(t)xy}I_{\alpha+1}(a(t)xy)e^{-a(t)xy}\Big)\\
&\quad-xyW_t(x,y)\partial_x\Big(\sqrt{2\pi a(t)xy}I_{\alpha+1}(a(t)xy)e^{-a(t)xy}\Big)\\
&\quad+yW_t(x,y)\Big(1-\sqrt{2\pi a(t)xy}I_{\alpha+1}(a(t)xy)e^{-a(t)xy}\Big)\\
&\quad+y^2\partial_yW_t(x,y)\Big(1-\sqrt{2\pi a(t)xy}I_{\alpha+1}(a(t)xy)e^{-a(t)xy}\Big)\\
&\quad-y^2W_t(x,y)\partial_y\Big(\sqrt{2\pi a(t)xy}I_{\alpha+1}(a(t)xy)e^{-a(t)xy}\Big)\Big]\\
&=\sum_{j=1}^{10}H_j(t,x,y),\quad t,x,y\in(0,\infty).
\end{aligned}$$

Note that

$$\begin{aligned}
&x\Big[\Big(\partial_xR\Big)(\sqrt{u}x,\sqrt{u}y;1)-\Big(\partial_xR_\alpha\Big)(\sqrt{u}x,\sqrt{u}y;1)\Big]+y\Big[\Big(\partial_yR\Big)(\sqrt{u}x,\sqrt{u}y;1)-\Big(\partial_yR_\alpha\Big)(\sqrt{u}x,\sqrt{u}y;1)\Big]\\
&\quad=\frac{x}{\sqrt{\pi}}\int_0^\infty\Big[\Big(\partial_xG_t\Big)(\sqrt{u}x,\sqrt{u}y)-\Big(\partial_xG_t^\alpha\Big)(\sqrt{u}x,\sqrt{u}y)\Big]\frac{dt}{\sqrt{t}}\\
&\qquad+\frac{y}{\sqrt{\pi}}\int_0^\infty\Big[\Big(\partial_yG_t\Big)(\sqrt{u}x,\sqrt{u}y)-\Big(\partial_yG_t^\alpha\Big)(\sqrt{u}x,\sqrt{u}y)\Big]\frac{dt}{\sqrt{t}},\quad x,y\in(0,\infty),\ x\neq y.
\end{aligned}$$

To estimate these integrals we split them in two parts as follows: for every $x,y\in(0,\infty)$, $x\neq y$, we consider

$$\int_0^\infty=\int_{\{t>0:ua(t)xy\le 1\}}+\int_{\{t>0:ua(t)xy>1\}}.$$

We firstly study the integral extended over $A_u(x,y)=\{t\in(0,\infty):ua(t)xy\le 1\}$. According to (23) we have that

$$\begin{aligned}
\frac{d}{dz}\Big[\sqrt{2\pi z}I_\alpha(z)e^{-z}\Big]&=\sqrt{2\pi}\frac{d}{dz}\Big[z^{\alpha+1/2}z^{-\alpha}I_\alpha(z)e^{-z}\Big]\\
&=\frac{\alpha+1/2}{z}\sqrt{2\pi z}I_\alpha(z)e^{-z}+\sqrt{2\pi z}I_{\alpha+1}(z)e^{-z}-\sqrt{2\pi z}I_\alpha(z)e^{-z},\quad z\in(0,\infty).
\end{aligned}\tag{27}$$

Then, by using (25) we obtain,

$$\left|\frac{d}{dz}\Big[\sqrt{2\pi z}I_\alpha(z)e^{-z}\Big]\right|\le Cz^{\alpha-1/2},\quad z\in(0,1).\tag{28}$$

Inequalities (25) and (28) lead to

$$\begin{aligned}
&\Big|x\partial_x\Big(G_t(x,y)-G_t^\alpha(x,y)\Big)+y\partial_y\Big(G_t(x,y)-G_t^\alpha(x,y)\Big)\Big|\\
&\quad\le Ca(t)\Big\{(x+y)W_t(x,y)+x^2|\partial_xW_t(x,y)|+y^2|\partial_yW_t(x,y)|\\
&\qquad+xy(|\partial_xW_t(x,y)|+|\partial_yW_t(x,y)|)\Big\},\quad t\in A_1(x,y),\ x,y\in(0,\infty),
\end{aligned}$$

because $\alpha\ge 1/2$ and $|1-2b(t)|\le a(t)$, $t\in(0,\infty)$. Since $a(t)t\le 1/2$ and $b(t)t\ge 1/4$, $t\in(0,\infty)$, for each $t\in A_1(x,y)$, $x,y\in(0,\infty)$, we have that

$$W_t(x,y)=\frac{1}{\sqrt{2\pi}}\sqrt{a(t)}e^{-b(t)(x^2+y^2)}e^{a(t)xy}\le C\frac{e^{-c(x^2+y^2)/t}}{\sqrt{t}},$$

and

$$x\Big|\partial_x W_t(x,y)\Big| + y\Big|\partial_y W_t(x,y)\Big| \leq C\frac{e^{-c(x^2+y^2)/t}}{\sqrt{t}}.$$

Thus, we obtain

$$\begin{aligned}\Big|x\partial_x\Big(G_t(x,y)-G_t^\alpha(x,y)\Big)+y\partial_y\Big(G_t(x,y)-G_t^\alpha(x,y)\Big)\Big| &\leq Ca(t)(x+y)\frac{e^{-c(x^2+y^2)/t}}{\sqrt{t}}\\ \leq Ca(t)e^{-c(x^2+y^2)/t} \leq C\frac{e^{-c(x^2+y^2)/t}}{t}, &\quad t\in A_1(x,y),\ x,y\in(0,\infty).\end{aligned}$$

Hence,

$$\begin{aligned}&\int_{t\in A_u(x,y)}\Big|x\partial_x(G_t-G_t^\alpha)(\sqrt{u}x,\sqrt{u}y)+y\partial_y(G_t-G_t^\alpha)(\sqrt{u}x,\sqrt{u}y)\Big|\frac{dt}{\sqrt{t}}\\ &\qquad\leq\frac{C}{\sqrt{u}}\int_0^\infty\frac{e^{-cu(x^2+y^2)/t}}{t^{3/2}}dt\leq\frac{C}{u\sqrt{x^2+y^2}},\quad x,y\in(0,\infty).\end{aligned}\tag{29}$$

We now estimate the integral extended to $B_u(x,y)=\{t\in(0,\infty): ua(t)xy>1\}$. This part is more involved than the previous one.

From (26) and (27) we deduce that, for every $n=2,3,\ldots$,

$$\begin{aligned}\frac{d}{dz}\Big(\sqrt{2\pi z}I_\alpha(z)e^{-z}\Big)&=\sqrt{2\pi}\Big(\frac{\alpha+1/2}{z}z^{1/2}I_\alpha(z)e^{-z}+z^{1/2}I_{\alpha+1}(z)e^{-z}-z^{1/2}I_\alpha(z)e^{-z}\Big)\\ &=\frac{\alpha+1/2}{z}\Big(\sum_{r=0}^n(-1)^r\frac{[\alpha,r]}{(2z)^r}+\mathcal{O}\Big(\frac{1}{z^{n+1}}\Big)\Big)+\sum_{r=0}^n(-1)^r\frac{[\alpha+1,r]}{(2z)^r}+\mathcal{O}\Big(\frac{1}{z^{n+1}}\Big)\\ &\quad-\sum_{r=0}^n(-1)^r\frac{[\alpha,r]}{(2z)^r}+\mathcal{O}\Big(\frac{1}{z^{n+1}}\Big)\\ &=\sum_{r=1}^n(-1)^r\frac{-(2\alpha+1)[\alpha,r-1]+[\alpha+1,r]-[\alpha,r]}{(2z)^r}+\mathcal{O}\Big(\frac{1}{z^{n+1}}\Big),\quad z\in(0,\infty).\end{aligned}$$

Since,

$$-(2\alpha+1)[\alpha,r-1]+[\alpha+1,r]-[\alpha,r]=2(r-1)[\alpha,r-1],\quad r=1,2,3,\ldots,$$

we get

$$\frac{d}{dz}\Big(\sqrt{2\pi z}I_\alpha(z)e^{-z}\Big)=\sum_{r=2}^n(-1)^r\frac{2(r-1)[\alpha,r-1]}{(2z)^r}+\mathcal{O}\Big(\frac{1}{z^{n+1}}\Big),\quad z\in(0,\infty),$$

for every $n=2,3,\ldots$. Then,

$$\frac{d}{dz}\Big(\sqrt{2\pi z}I_\alpha(z)e^{-z}\Big)=\mathcal{O}\Big(\frac{1}{z^2}\Big),\quad z\in(0,\infty).\tag{30}$$

By using (26) we obtain

$$\begin{aligned}&\int_{t\in B_u(x,y)}\Big|H_1(t,\sqrt{u}x,\sqrt{u}y)+H_8(t,\sqrt{u}x,\sqrt{u}y)\Big|\frac{dt}{\sqrt{t}}\leq C\frac{x+y}{\sqrt{uxy}}\int_{t\in B_u(x,y)}W_t(\sqrt{u}x,\sqrt{u}y)\frac{dt}{\sqrt{t}}\\ &\leq C\frac{x+y}{\sqrt{uxy}}\int_{t\in B_u(x,y)}\sqrt{a(t)}e^{-b(t)u|x-y|^2}\frac{dt}{\sqrt{t}}\leq C\frac{x+y}{u^{1/4}(xy)^{3/4}}\int_{t\in B_u(x,y)}\frac{\sqrt{a(t)}}{(uxy)^{1/4}}e^{-cu|x-y|^2/t}\frac{dt}{\sqrt{t}}\\ &\leq C\frac{x+y}{u^{1/4}(xy)^{3/4}}\int_0^\infty\frac{e^{-cu|x-y|^2/t}}{t^{5/4}}dt\leq C\frac{x+y}{u^{1/2}(xy)^{3/4}|x-y|^{1/2}},\quad x,y\in(0,\infty),\ x\neq y.\end{aligned}\tag{31}$$

Asymptotic behavior (26) leads also to

$$\begin{aligned}|H_2(t,x,y)+H_4(t,x,y)|&\leq C\frac{a(t)x}{(a(t)xy)^{3/4}}\Big|x\partial_x W_t(x,y)+y\partial_y W_t(x,y)\Big|\\ &=C\frac{a(t)^{3/4}x^{1/4}}{y^{3/4}}[b(t)(x^2+y^2)-a(t)xy]e^{-b(t)(x^2+y^2)+a(t)xy}\\ &\leq C\frac{x^{1/4}}{t^{3/4}y^{3/4}}e^{-c|x-y|^2/t}\quad x,y\in(0,\infty),\ t\in B_1(x,y).\end{aligned}$$

We obtain

(32)
$$\int_{t\in B_u(x,y)} \Big|H_2(t,\sqrt{u}x,\sqrt{u}y)+H_4(t,\sqrt{u}x,\sqrt{u}y)\Big|\frac{dt}{\sqrt{t}} \leq C\frac{x^{1/4}}{u^{1/2}y^{3/4}|x-y|^{1/2}},\ x,y\in(0,\infty),\ x\neq y.$$

Also, we have that

$$\begin{aligned}|H_6(t,x,y)+H_9(t,x,y)| &\leq C\frac{a(t)y}{(a(t)xy)^{3/4}}\Big|x\partial_x W_t(x,y)+y\partial_y W_t(x,y)\Big| \\ &\leq C\frac{y^{1/4}}{t^{3/4}x^{3/4}}e^{-c|x-y|^2/t} \quad x,y\in(0,\infty),\ t\in B_1(x,y).\end{aligned}$$

Then

(33)
$$\int_{t\in B_u(x,y)} \Big|H_6(t,\sqrt{u}x,\sqrt{u}y)+H_9(t,\sqrt{u}x,\sqrt{u}y)\Big|\frac{dt}{\sqrt{t}} \leq C\frac{y^{1/4}}{u^{1/2}x^{3/4}|x-y|^{1/2}},\ x,y\in(0,\infty),\ x\neq y.$$

According to (30) we obtain

$$\Big|H_3(t,x,y)+H_5(t,x,y)\Big| \leq C\frac{a(t)^2x^2y}{(a(t)xy)^{7/4}}W_t(x,y) \leq C\frac{x^{1/4}}{t^{3/4}y^{3/4}}e^{-c|x-y|^2/t},\ x,y\in(0,\infty),\ t\in B_1(x,y).$$

Hence,

(34)
$$\int_{t\in B_u(x,y)} \Big|H_3(t,\sqrt{u}x,\sqrt{u}y)+H_5(t,\sqrt{u}x,\sqrt{u}y)\Big|\frac{dt}{\sqrt{t}} \leq C\frac{x^{1/4}}{u^{1/2}y^{3/4}|x-y|^{1/2}},\ x,y\in(0,\infty),\ x\neq y.$$

Similarly we can show that

(35)
$$\int_{t\in B_u(x,y)} \Big|H_7(t,\sqrt{u}x,\sqrt{u}y)+H_{10}(\sqrt{u}x,\sqrt{u}y,t)\Big|\frac{dt}{\sqrt{t}} \leq C\frac{y^{1/4}}{u^{1/2}x^{3/4}|x-y|^{1/2}},\ x,y\in(0,\infty),\ x\neq y.$$

By putting together (31)–(35) we get

$$\begin{aligned}&\int_{t\in B_u(x,y)} \Big|x\partial_x\Big(G_t-G_t^\alpha\Big)(\sqrt{u}x,\sqrt{u}y)+y\partial_y\Big(G_t-G_t^\alpha\Big)(\sqrt{u}x,\sqrt{u}y)\Big|\frac{dt}{\sqrt{t}} \\ &\leq \frac{C}{u}\frac{1}{|x-y|^{1/2}}\Big(\frac{x^{1/4}}{y^{3/4}}+\frac{y^{1/4}}{x^{3/4}}\Big), \quad x,y\in(0,\infty),\ x\neq y. \qquad (36)\end{aligned}$$

From (29) and (36) it follows that

$$\begin{aligned}&\int_0^\infty \Big|x\partial_x\Big(G_t-G_t^\alpha\Big)(\sqrt{u}x,\sqrt{u}y)+y\partial_y\Big(G_t-G_t^\alpha\Big)(\sqrt{u}x,\sqrt{u}y)\Big|\frac{dt}{\sqrt{t}} \\ &\leq \frac{C}{u}\Big[\frac{1}{|x-y|^{1/2}}\Big(\frac{x^{1/4}}{y^{3/4}}+\frac{y^{1/4}}{x^{3/4}}\Big)+\frac{1}{\sqrt{x^2+y^2}}\Big], \quad x,y\in(0,\infty),\ x\neq y. \qquad (37)\end{aligned}$$

In order to get the same estimate as the one shown in (21) for $|\partial_u(R_\alpha(x,y;u)-R(x,y;u))|$, and following a similar proceeding, we can obtain for every $0<y<x/2$ or $0<2x<y$

$$|H_j(t,x,y)| \leq C\frac{\max\{x,y\}}{t^{3/2}}e^{-c|x-y|^2/t}, \quad t\in B_1(x,y),\ j=1,\dots,10,$$

and together with (29) we get

$$\begin{aligned}&\int_0^\infty \Big|x\partial_x\Big(G_t-G_t^\alpha\Big)(\sqrt{u}x,\sqrt{u}y)+y\partial_y\Big(G_t-G_t^\alpha\Big)(\sqrt{u}x,\sqrt{u}y)\Big|\frac{dt}{\sqrt{t}} \\ &\leq \frac{C}{u}\left\{\begin{array}{ll} 1/x, & 0<y<x/2<\infty, \\ 1/y, & 0<2x<y<\infty. \end{array}\right. \qquad (38)\end{aligned}$$

By combining (21), (22), (37) and (38) we conclude that

$$\left|\partial_u\Big(R_\alpha(x,y;u)-R(x,y;u)\Big)\right| \le \frac{C}{u}\begin{cases} \dfrac{1}{x}, & 0<y<x/2<\infty,\\[2mm] \dfrac{1}{x}\Big(1+\sqrt{\dfrac{x}{|x-y|}}\Big), & 0<x/2<y<2x<\infty,\\[2mm] \dfrac{1}{y}, & 0<2x<y<\infty. \end{cases} \tag{39}$$

Here, C does not depend on u.

By differentiating in (22) we get

$$\begin{aligned}\partial_u^2\Big(R_\alpha(x,y;u)-R(x,y;u)\Big) &= \frac{1}{4u^{3/2}}\Big(R_\alpha-R\Big)(\sqrt{u}x,\sqrt{u}y;1)\\ &+\frac{1}{4u}\Big[x\partial_x\Big(R_\alpha-R\Big)(\sqrt{u}x,\sqrt{u}y;1)+y\partial_y\Big(R_\alpha-R\Big)(\sqrt{u}x,\sqrt{u}y;1)\Big]\\ &+\frac{1}{4\sqrt{u}}\Big[x^2\partial_x^2\Big(R_\alpha-R\Big)(\sqrt{u}x,\sqrt{u}y;1)+2xy\partial_{xy}^2\Big(R_\alpha-R\Big)(\sqrt{u}x,\sqrt{u}y;1)\\ &+y^2\partial_y^2\Big(R_\alpha-R\Big)(\sqrt{u}x,\sqrt{u}y;1)\Big],\quad x,y\in(0,\infty).\end{aligned}$$

Then, by proceeding as in the previous case we obtain

$$\left|\partial_u^2\Big(R_\alpha(x,y;u)-R(x,y;u)\Big)\right| \le \frac{C}{u^{3/2}}\begin{cases} \dfrac{1}{x}, & 0<y<x/2<\infty,\\[2mm] \dfrac{1}{x}\Big(1+\sqrt{\dfrac{x}{|x-y|}}\Big), & 0<x/2<y<2x<\infty,\\[2mm] \dfrac{1}{y}, & 0<2x<y<\infty. \end{cases} \tag{40}$$

We consider the Hardy operators I_0 and I^∞ defined by

$$I_0(g)(x)=\frac{1}{x}\int_0^x g(y)dy,\quad \text{and}\quad I^\infty(g)(x)=\int_x^\infty \frac{g(y)}{y}dy,\quad x\in(0,\infty).$$

It is well-known [13, p. 244, (9.9.1) and (9.9.2)] that I_0 and I^∞ define bounded operators from $L^p(0,\infty)$ into itself. On the other hand, Jensen's inequality allows us to see that the operator I given by

$$I(g)(x)=\frac{1}{x}\int_{x/2}^{2x}\Big(1+\sqrt{\frac{x}{|x-y|}}\Big)g(y)dy,\quad x\in(0,\infty),$$

is bounded from $L^p(0,\infty)$ into itself.

By (39) we deduce that the operator

$$T_\alpha(u)(g)(x)=\int_0^\infty \partial_u\Big(R_\alpha(x,y;u)-R(x,y;u)\Big)g(y)dy,\quad x\in(0,\infty),$$

is bounded from $L^p(0,\infty)$ into itself.

We can write, for every $g\in L^p(0,\infty)$,

$$\begin{aligned}&\frac{1}{h}\Big[R_\alpha(u+h)(g)(x)-R(u+h)(g)(x)-\Big(R_\alpha(u)(g)(x)-R(u)(g)(x)\Big)\Big]-T_\alpha(u)(g)(x)\\ &=\int_0^\infty\Big[\frac{1}{h}\int_u^{u+h}\partial_\lambda\Big(R_\alpha(x,y;\lambda)-R(x,y;\lambda)\Big)d\lambda-\partial_u\Big(R_\alpha(x,y;u)-R(x,y;u)\Big)\Big]g(y)dy\\ &=\int_0^\infty\frac{1}{h}\int_u^{u+h}\int_u^\lambda\partial_z^2\Big(R_\alpha(x,y;z)-R(x,y;z)\Big)dzd\lambda g(y)dy,\quad 0<|h|<u,\ x\in(0,\infty).\end{aligned}$$

According to (40) it follows that

$$\begin{aligned}\left\|\frac{(R_\alpha-R)(u+h)(g)-(R_\alpha-R)(u)(g)}{h}-T_\alpha(u)(g)\right\|_{L^p(0,\infty)} &\le C\left|\frac{1}{h}\int_u^{u+h}\int_u^\lambda\frac{dzd\lambda}{z^{3/2}}\right|\|g\|_{L^p(0,\infty)}\\ &\le C\left|\frac{h-2\sqrt{u^2+uh}+2u}{\sqrt{u}h}\right|\|g\|_{L^p(0,\infty)},\quad 0<|h|<u.\end{aligned}$$

Hence,

$$\lim_{h\to 0^+}\frac{(R_\alpha - R)(u+h)-(R_\alpha - R)(u)}{h}=T_\alpha(u),$$

in the sense of convergence in $\mathcal{L}(L^p(0,\infty))$.

We can proceed in a similar way when $u<0$. We conclude that the function

$$\begin{array}{rcl}\mathbb{R}\setminus\{0\} & \longrightarrow & \mathcal{L}(L^p(0,\infty))\\ u & \longmapsto & R_\alpha(u)\end{array}$$

is differentiable and that, for every $g\in L^p(0,\infty)$,

$$\partial_u R_\alpha(u)g(x)=\lim_{\varepsilon\to 0^+}\int_{|x-y|>\varepsilon}\partial_u R_\alpha(x,y;u)g(u)du,\quad \text{a.e. } x\in(0,\infty).$$

□

Lemma 3.3. *Let $\alpha\geq 1/2$. The family of operators $\{u\frac{d}{du}R_\alpha(u)\}_{u\in\mathbb{R}\setminus\{0\}}$ is R-bounded in $L^p(0,\infty)$.*

Proof. From (39) we deduce that

$$\sup_{u\in\mathbb{R}\setminus\{0\}}\|u\partial_u R_\alpha(u)\|_{\mathcal{L}(L^2(0,\infty))}<\infty.$$

and

$$\sup_{u\in\mathbb{R}\setminus\{0\}}\left|u\partial_u\Big(R_\alpha(x,y;u)-R(x,y;u)\Big)\right|\leq\frac{C}{|x-y|},\quad x,y\in(0,\infty),x\neq y. \tag{41}$$

Assume that $u>0$. We can write

$$\begin{aligned}\partial_x\Big[u\partial_u\Big(R_\alpha(x,y;u)-R(x,y;u)\Big)\Big]&=\partial_x\Big[u\partial_u\Big(\sqrt{u}R_\alpha(\sqrt{u}x,\sqrt{u}y;1)-\sqrt{u}R(\sqrt{u}x,\sqrt{u}y;1)\Big)\Big]\\ &=u\partial_u\Big[u(\partial_x R_\alpha)(\sqrt{u}x,\sqrt{u}y;1)-u(\partial_x R)(\sqrt{u}x,\sqrt{u}y;1)\Big]\\ &=u\Big[(\partial_x R_\alpha)(\sqrt{u}x,\sqrt{u}y;1)-(\partial_x R)(\sqrt{u}x,\sqrt{u}y;1)\Big]\\ &\quad+\frac{u^{3/2}}{2}\Big[x(\partial_x^2 R_\alpha)(\sqrt{u}x,\sqrt{u}y;1)-x(\partial_x^2 R)(\sqrt{u}x,\sqrt{u}y;1)\\ &\quad+y(\partial_{xy}^2 R_\alpha)(\sqrt{u}x,\sqrt{u}y;1)-y(\partial_{xy}^2 R)(\sqrt{u}x,\sqrt{u}y;1)\Big],\quad x,y\in(0,\infty),\ x\neq y.\end{aligned}$$

Moreover,

$$\begin{aligned}\partial_x^2\Big[G_t(x,y)-G_t^\alpha(x,y)\Big]&=(1-2b(t))\Big\{2\partial_x\Big[W_t(x,y)\Big(1-\sqrt{2\pi a(t)xy}I_\alpha(a(t)xy)e^{-a(t)xy}\Big)\Big]\\ &\quad+x\Big[\partial_x^2 W_t(x,y)\Big(1-\sqrt{2\pi a(t)xy}I_\alpha(a(t)xy)e^{-a(t)xy}\Big)\\ &\quad-2\partial_x W_t(x,y)\partial_x\Big(\sqrt{2\pi a(t)xy}I_\alpha(a(t)xy)e^{-a(t)xy}\Big)\\ &\quad-W_t(x,y)\partial_x^2\Big(\sqrt{2\pi a(t)xy}I_\alpha(a(t)xy)e^{-a(t)xy}\Big)\Big]\Big\}\\ &\quad+a(t)y\Big\{\partial_x^2 W_t(x,y)\Big(1-\sqrt{2\pi a(t)xy}I_{\alpha+1}(a(t)xy)e^{-a(t)xy}\Big)\\ &\quad-2\partial_x W_t(x,y)\partial_x\Big(\sqrt{2\pi a(t)xy}I_{\alpha+1}(a(t)xy)e^{-a(t)xy}\Big)\\ &\quad-W_t(x,y)\partial_x^2\Big(\sqrt{2\pi a(t)xy}I_{\alpha+1}(a(t)xy)e^{-a(t)xy}\Big)\Big\},\quad t,x,y\in(0,\infty).\end{aligned}\tag{42}$$

By using (23) and (24) we get

$$\frac{d^2}{dz^2}\Big[\sqrt{z}I_\alpha(z)e^{-z}\Big]=e^{-z}\Big[\Big(\frac{4\alpha^2-1}{4z^2}+2-\frac{2\alpha+1}{z}\Big)\sqrt{z}I_\alpha(z)-2\sqrt{z}I_{\alpha+1}(z)\Big],\quad z\in(0,\infty).\tag{43}$$

According to (26) it follows that

$$\frac{d^2}{dz^2}\Big[\sqrt{z}I_\alpha(z)e^{-z}\Big]=\mathcal{O}\Big(\frac{1}{z^3}\Big),\quad z\in(0,\infty).\tag{44}$$

Equalities (26), (30), (42) and (44) lead to

$$\begin{aligned}x\Big|\partial_x^2\Big(G_t-G_t^\alpha\Big)(x,y)\Big| &\le Cxa(t)\Big\{|\partial_x W_t(x,y)|\frac{1}{(a(t)xy)^{1/2}}+W_t(x,y)\frac{a(t)y}{a(t)xy}\\&\quad+x|\partial_x^2 W_t(x,y)|\frac{1}{a(t)xy}+x|\partial_x W_t(x,y)|\frac{a(t)y}{(a(t)xy)^{3/2}}+xW_t(x,y)\frac{(a(t)y)^2}{(a(t)xy)^2}\\&\quad+y|\partial_x^2 W_t(x,y)|\frac{1}{a(t)xy}+y|\partial_x W_t(x,y)|\frac{a(t)y}{(a(t)xy)^{3/2}}+yW_t(x,y)\frac{(a(t)y)^2}{(a(t)xy)^2}\Big\}\\&\le C\frac{e^{-c|x-y|^2/t}}{t^{3/2}},\quad t\in B_1(x,y),\ 0<x/2<y<2x,\end{aligned}$$

because

$$\Big|\partial_x^\ell W_t(x,y)\Big|\le C\frac{e^{-c|x-y|^2/t}}{t^{(\ell+1)/2}},\quad t,x,y\in(0,\infty),\ \ell=0,1,2.$$

Analogously, when $0<y<x/2$ or $2x<y<\infty$, we get

$$\begin{aligned}x\Big|\partial_x^2\Big(G_t-G_t^\alpha\Big)(x,y)\Big| \le& Cxa(t)\Big\{|\partial_x W_t(x,y)|+W_t(x,y)a(t)y+x|\partial_x^2 W_t(x,y)|\\&+x|\partial_x W_t(x,y)|a(t)y+xW_t(x,y)(a(t)y)^2+y|\partial_x^2 W_t(x,y)|\\&+y|\partial_x W_t(x,y)|a(t)y+yW_t(x,y)(a(t)y)^2\Big\}\\\le& C\frac{x}{t}\Big\{\frac{1}{t}+\frac{x+y}{t^{3/2}}+\frac{xy+y^2}{t^2}+\frac{xy^2+y^3}{t^{5/2}}\Big\}e^{-c|x-y|^2/t}\\\le& C\frac{e^{-c|x-y|^2/t}}{t^{3/2}},\quad t\in B_1(x,y).\end{aligned}$$

Hence, we obtain for each $x,y\in(0,\infty)$, $x\neq y$,

$$x\int_{t\in B_u(x,y)}\Big|\partial_x^2\Big(G_t-G_t^\alpha\Big)(\sqrt{u}x,\sqrt{u}y)\Big|\frac{dt}{\sqrt{t}}\le\frac{C}{\sqrt{u}}\int_0^\infty\frac{e^{-cu|x-y|^2/t}}{t^2}dt\le\frac{C}{u^{3/2}|x-y|^2}. \tag{45}$$

On the other hand, since $\alpha\ge 1/2$, from (25), (28) and (43) we deduce, for every $0<z\le 1$,

$$\Big|\frac{d^\ell}{dz^\ell}\Big(\sqrt{z}I_\alpha(z)e^{-z}\Big)\Big|\le C,\quad \ell=0,1,\qquad \Big|\frac{d^2}{dz^2}\Big(\sqrt{z}I_\alpha(z)e^{-z}\Big)\Big|\le\frac{C}{z}.$$

Hence,

$$\begin{aligned}x\Big|\partial_x^2\Big(G_t-G_t^\alpha\Big)(x,y)\Big|&\le Ca(t)\Big\{(1+a(t)y^2)W_t(x,y)+(x+y)|\partial_x W_t(x,y)|+(x^2+y^2)|\partial_x^2 W_t(x,y)|\Big\}\\&\le C\frac{e^{-c(x^2+y^2)/t}}{t^{3/2}},\quad t\in A_1(x,y),\ x,y\in(0,\infty).\end{aligned}$$

Then, for every $x,y\in(0,\infty)$,

$$\int_{t\in A_u(x,y)}x\Big|\partial_x^2\Big(G_t-G_t^\alpha\Big)(\sqrt{u}x,\sqrt{u}y)\Big|\frac{dt}{\sqrt{t}}\le\frac{C}{\sqrt{u}}\int_0^\infty\frac{e^{-cu(x^2+y^2)/t}}{t^2}dt\le\frac{C}{u^{3/2}(x^2+y^2)}. \tag{46}$$

From (45) and (46) it follows that

$$u^{3/2}x\Big|(\partial_x R_\alpha)(\sqrt{u}x,\sqrt{u}y;1)-(\partial_x R)(\sqrt{u}x,\sqrt{u}y;1)\Big|\le\frac{C}{|x-y|^2},\quad x,y\in(0,\infty),\ x\neq y,$$

where C does not depend on u.

In a similar way we can see that

$$u^{3/2}y\Big|(\partial_{xy}^2 R_\alpha)(\sqrt{u}x,\sqrt{u}y;1)-(\partial_{xy}^2 R)(\sqrt{u}x,\sqrt{u}y;1)\Big|\le\frac{C}{|x-y|^2},\quad x,y\in(0,\infty),\ x\neq y,$$

with $C>0$ independent of u.

Moreover by (17) and [27, Theorem 3.3] we have that

$$u\Big|(\partial_x R_\alpha)(\sqrt{u}x,\sqrt{u}y;1)-(\partial_x R)(\sqrt{u}x,\sqrt{u}y;1)\Big|\le\frac{C}{|x-y|^2},\quad x,y\in(0,\infty),\ x\neq y,$$

where C does not depend on u. Hence, we conclude that

$$\Big|\partial_x\Big(u\partial_u\Big[R_\alpha(x,y;u)-R(x,y;u)\Big]\Big)\Big|\le\frac{C}{|x-y|^2},\quad u,x,y\in(0,\infty),\ x\neq y.$$

Analogously, we can show that

$$\Big|\partial_y\Big(u\partial_u\Big[R_\alpha(x,y;u)-R(x,y;u)\Big]\Big)\Big|\leq\frac{C}{|x-y|^2},\quad u,x,y\in(0,\infty),\ x\neq y.$$

When $u<0$ we can proceed similarly, and we obtain, for every $x,y\in(0,\infty)$, $x\neq y$,

$$\sup_{u\in\mathbb{R}\setminus\{0\}}\Big\{\Big|\partial_x\Big(u\partial_u\Big[R_\alpha(x,y;u)-R(x,y;u)\Big]\Big)\Big|+\Big|\partial_y\Big(u\partial_u\Big[R_\alpha(x,y;u)-R(x,y;u)\Big]\Big)\Big|\Big\}\leq\frac{C}{|x-y|^2}.$$

According to [15, Proof of Lemma 2.2 and Proposition 2.3], we have that

$$\sup_{u\in\mathbb{R}\setminus\{0\}}\Big|u\partial_uR(x,y;u)\Big|\leq\frac{C}{|x-y|},\quad x,y\in(0,\infty),\ x\neq y,$$

and

$$\sup_{u\in\mathbb{R}\setminus\{0\}}\Big\{\Big|\partial_x\Big(u\partial_uR(x,y;u)\Big)\Big|+\Big|\partial_y\Big(u\partial_uR(x,y;u)\Big)\Big|\Big\}\leq\frac{C}{|x-y|^2},\quad x,y\in(0,\infty),\ x\neq y.$$

Then, from (41) and (46), we conclude that

$$\sup_{u\in\mathbb{R}\setminus\{0\}}\Big|u\partial_uR_\alpha(x,y;u)\Big|\leq\frac{C}{|x-y|},\quad x,y\in(0,\infty),\ x\neq y,$$

and

$$\sup_{u\in\mathbb{R}\setminus\{0\}}\Big\{\Big|\partial_x\Big(u\partial_uR_\alpha(x,y;u)\Big)\Big|+\Big|\partial_y\Big(u\partial_uR_\alpha(x,y;u)\Big)\Big|\Big\}\leq\frac{C}{|x-y|^2},\quad x,y\in(0,\infty),\ x\neq y.$$

□

References

[1] J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Springer-Verlag, Berlin, 1976.

[2] J. J. Betancor, J. C. Fariña, L. Rodríguez-Mesa, and A. Sanabria, *Higher order Riesz transforms for Laguerre expansions*, Illinois J. of Math., 55 (2011), pp. 27–68.

[3] B. Bongioanni and J. L. Torrea, *Sobolev spaces associated to the harmonic oscillator*, Proc. Indian Acad. Sci. Math. Sci., 116 (2006), pp. 337–360.

[4] ———, *What is a Sobolev space for the Laguerre function systems?*, Studia Math., 192 (2009), pp. 147–172.

[5] P. Chen and A. Sikora, *Sharp spectral multipliers for a new class of Grushin type operators*. Preprint 2012 (`arXiv:1210.0322v1`).

[6] M. Christ, *L^p bounds for spectral multipliers on nilpotent groups*, Trans. Amer. Math. Soc., 328 (1991), pp. 73–81.

[7] R. R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Mathematics, Vol. 242, Springer-Verlag, Berlin, 1971.

[8] X. T. Duong, E. M. Ouhabaz, and A. Sikora, *Plancherel-type estimates and sharp spectral multipliers*, J. Funct. Anal., 196 (2002), pp. 443–485.

[9] X. T. Duong, A. Sikora, and L. Yan, *Weighted norm inequalities, Gaussian bounds and sharp spectral multipliers*, J. Funct. Anal., 260 (2011), pp. 1106–1131.

[10] K. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, vol. 194 of Graduate Texts in Mathematics, Springer-Verlag, New York, 2000.

[11] G. B. Folland and E. M. Stein, *Hardy spaces on homogeneous groups*, vol. 28 of Mathematical Notes, Princeton University Press, Princeton, N.J., 1982.

[12] E. Harboure, J. L. Torrea, and B. E. Viviani, *Riesz transforms for Laguerre expansions*, Indiana Univ. Math. J., 55 (2006), pp. 999–1014.

[13] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, Cambridge, at the University Press, 1934.

[14] W. Hebisch, *A multiplier theorem for Schrödinger operators*, Colloq. Math., 60/61 (1990), pp. 659–664.

[15] K. Jotsaroop, P. Sanjay, and S. Thangavelu, *Riesz transforms and multipliers for the Grushin operator*. To appear in J. Analyse Math. (`arXiv:1110.3227v1`).

[16] N. N. Lebedev, *Special functions and their applications*, Dover Publications Inc., New York, 1972.

[17] C. Martínez Carracedo and M. Sanz Alix, *The theory of fractional powers of operators*, vol. 187 of North-Holland Mathematics Studies, North-Holland Publishing Co., Amsterdam, 2001.

[18] A. Martini and D. Müller, *A sharp multiplier theorem for Grushin operators in arbitrary dimensions*. Preprint 2012 (`arXiv:1210.3564v1`).

[19] A. Martini and A. Sikora, *Weighted Plancherel estimates and sharp spectral multipliers for the Grushin operators*. To appear in Math. Res. Lett. (`arXiv:1204.1159v1`).

[20] S. Meda, *A general multiplier theorem*, Proc. Amer. Math. Soc., 110 (1990), pp. 639–647.

[21] B. Muckenhoupt and D. W. Webb, *Two-weight norm inequalities for the Cesàro means of Hermite expansions*, Trans. Amer. Math. Soc., 354 (2002), pp. 4525–4537 (electronic).

[22] A. Nowak, *Heat-diffusion and Poisson integrals for Laguerre and special Hermite expansions on weighted L^p spaces*, Studia Math., 158 (2003), pp. 239–268.

[23] A. Nowak and K. Stempak, *Riesz transforms and conjugacy for Laguerre function expansions of Hermite type*, J. Funct. Anal., 244 (2007), pp. 399–443.

[24] ———, *On L^p-contractivity of Laguerre semigroups*. To appear in Illinois J. of Math. (arXiv:1011.5437v1).
[25] E. M. Ouhabaz, *Gaussian estimates and holomorphy of semigroups*, Proc. Amer. Math. Soc., 123 (1995), pp. 1465–1474.
[26] A. Sikora, *Multivariable spectral multipliers and analysis of quasielliptic operators on fractals*, Indiana Univ. Math. J., 58 (2009), pp. 317–334.
[27] K. Stempak and J. L. Torrea, *Poisson integrals and Riesz transforms for Hermite function expansions with weights*, J. Funct. Anal., 202 (2003), pp. 443–472.
[28] G. Szegő, *Orthogonal polynomials*, American Mathematical Society, Providence, R.I., fourth ed., 1975. American Mathematical Society, Colloquium Publications, Vol. XXIII.
[29] S. Thangavelu, *Lectures on Hermite and Laguerre expansions*, vol. 42 of Mathematical Notes, Princeton University Press, Princeton, NJ, 1993.
[30] A. Torchinsky, *Real-variable methods in harmonic analysis*, vol. 123 of Pure and Applied Mathematics, Academic Press Inc., Orlando, FL, 1986.
[31] L. Weis, *Operator-valued Fourier multiplier theorems and maximal L_p-regularity*, Math. Ann., 319 (2001), pp. 735–758.
[32] A. Yagi, *Coïncidence entre des espaces d'interpolation et des domaines de puissances fractionnaires d'opérateurs*, C. R. Acad. Sci. Paris Sér. I Math., 299 (1984), pp. 173–176.

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